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CHARACTER TABLE AND BLOCKS OF FINITE SIMPLE TRIALITY GROUPS ${}^3D_4(q)$

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ABSTRACT. Based on recent work of Spaltenstein [**14**] and the Deligne-Lusztig theory of irreducible characters of finite groups of Lie type, in this paper the character table of the finite simple groups ${}^3D_4(q)$ is given. As an application we obtain a classification of the irreducible characters of ${}^3D_4(q)$ into r-blocks for all primes $r > 0$. This enables us to verify Brauer's height zero conjecture, his conjecture on the bound of irreducible characters belonging to a give block, and the Alperin-McKay conjecture for the simple triality groups ${}^3D_4(q)$. It also follows that for every prime r there are blocks of defect zero in ${}^3D_4(q)$.

Introduction. Let $G_\sigma = {}^3D_4(q)$ be a simple triality group defined over a finite field GF(q) with $q = p^n$ elements, where $p > 0$ is a prime number and n is a positive integer.

In [**14**] N. Spaltenstein computed the values of the eight unipotent irreducible characters of G_σ. Using his results we determine the character table of G_σ in §4. In Theorem 4.3 the nonunipotent irreducible characters of G_σ are presented in the form of precise linear combinations of the virtual Deligne-Lusztig characters $R_{T,\Theta}$, where Θ is a linear character of the σ-fixed points of a σ-stable maximal torus T of the corresponding algebraic group G. The values of the Deligne-Lusztig characters are given in Table 3.6.

By Lusztig's Jordan form of the irreducible characters of a finite group of Lie type [**11**] each irreducible character χ of G_σ is of the form $\chi = \chi_{t,u}$, where t is a semisimple element of G_σ and χ_u is a unipotent irreducible character of the centralizer $C_{G_\sigma}(t)$ of t. The group theoretical structure of the centralizers $C_{G_\sigma}(t)$ of the semisimple elements t of G_σ is given in Proposition 2.2, and of the 7 (up to G_σ-conjugacy) maximal tori T_i, $0 \leqslant i \leqslant 6$, in Proposition 1.2. It follows that $C_{G_\sigma}(t)$ has at most three unipotent irreducible characters, namely the trivial 1, the Steinberg character St or a unipotent character of degree either $qs = q(q+1)$ or $qs' = q(q-1)$. If $t \neq 1$ is regular, we write χ_t instead of $\chi_{t,1}$, in all other cases $\chi_{t,1}$, $\chi_{t,\mathrm{St}}$, $\chi_{t,qs}$, $\chi_{t,qs'}$ or $\chi_{t,\mathrm{St}\,\mathrm{St}'}$. A complete classification of the irreducible characters of G_σ with their degrees is given in Table 4.4.

On the set of conjugacy classes of semisimple elements t of G_σ one can define an equivalence relation as follows. Two such conjugacy classes $t_1^{G_\sigma}$ and $t_2^{G_\sigma}$ are equivalent if and only if their centralizers $C_{G_\sigma}(t_1)$ and $C_{G_\sigma}(t_2)$ are G_σ-conjugate. If q is odd, there are 15 equivalence classes with representatives s_i, $1 \leqslant i \leqslant 15$, where $s_1 = 1$

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and $s_2 \neq 1$ is the unique conjugacy class of involutions of G_σ. If q is even, the equivalence class of s_2 does not exist, and we have only 14 equivalence classes. Using the first author's work on the Brauer complex **[5]** of G_σ and the computer, we obtain in Table 4.4 the numbers of semisimple conjugacy classes of G_σ belonging to a given equivalence class $[s_i]$, $1 \leqslant i \leqslant 15$. Applying then Proposition 2.2 and Spaltenstein's characterization of the unipotent conjugacy classes of G_σ **[14]**, we can give in Proposition 2.3 a complete classification of all conjugacy classes of G_σ. In particular, we show that the number $k(G_\sigma)$ of all conjugacy classes of G_σ is

$$k(G_\sigma) = q^4 + q^3 + q^2 + q + 5, \quad \text{if } 2 \mid q, \text{ and}$$
$$k(G_\sigma) = q^4 + q^3 + q^2 + q + 6, \quad \text{if } 2 \nmid q.$$

By means of these results we determine in §5 the distribution of the irreducible characters of G_σ into r-blocks, where r is a prime number dividing the group order $|G_\sigma|$. If $r = p$, then by Humphrey's theorem **[10]** G_σ has only the principal p-block B_0 and a block B of defect zero consisting of the irreducible Steinberg character. For $r \neq p$ Theorem 5.9 asserts that each r-block B with defect group D determines, up to G_σ-conjugacy, a unique semisimple r'-element s of G_σ such that an irreducible character $\chi_{t,u}$ of G_σ belongs to B if and only if t is G_σ-conjugate to sy for some $y \in D$, and χ_u is an irreducible unipotent character of $C_{G_\sigma}(sy)$ such that $\widehat{sy}\chi_u$ belongs to an r-block $\tilde{B}$ of $C_{G_\sigma}(sy)$ with defect group D satisfying $B = \tilde{B}^G$. This result can be considered to be an analogue of the Fong-Srinivasan characterization **[8]** of the r-blocks of the general linear and unitary groups.

In Corollary 5.11 we show that for all primes $r > 0$ and all r-blocks B of G_σ with defect group $\delta(B) =_{G_\sigma} D$ the number of all irreducible characters of G_σ belonging to B is bounded by $k(B) \leqslant |D|$. This verifies a well-known conjecture of R. Brauer, see **[7]**, in the case of the simple triality groups. He also conjectured that an r-block B of a finite group G has only irreducible characters of height zero if and only if its defect group $\delta(B) =_G D$ is abelian. In case $G = G_\sigma$ this is shown for all primes r in Corollary 5.10.

Let $k_0(B)$ be the number of irreducible characters of an r-block B of G with height zero. If $\delta(B) =_G D$ denotes the defect group of D, $H = N_G(D)$, and B_1 is the Brauer correspondent of B in H, then the Alperin-McKay conjecture asserts that $k_0(B) = k_0(B_1)$. In the case of $G = G_\sigma$, we verify it for all primes r; see Corollary 5.12.

Another application of Table 4.4 yields that in G_σ there are r-blocks B of defect zero for every prime $r > 0$; see Corollary 5.1.

Concerning the notation and terminology we refer to the books by Carter **[2]**, Deriziotis **[4]**, Feit **[7]**, and Lusztig **[11]**.

1. Notations and known results on ${}^3D_4(q)$. Let G be a simple simply connected algebraic group of Dynkin diagram type D_4 over the algebraic closure K of the prime field $\mathrm{GF}(p) = \mathbf{F}_p$, $p > 0$. Let $q = p^m$ for some positive integer m, and let $\mathrm{GF}(q) = \mathbf{F}_q$ be the field with q elements. F^* denotes the multiplicative group of every field F.

Let T be a maximal torus of G, Φ the set of roots of G relative to T, $X = \mathrm{Hom}(T, K^*)$—the group of rational characters of T, $Y = \mathrm{Hom}(K^*, T)$—the group of one-parameter subgroups of T. On the real vector space $V = Y \otimes \mathbf{R}$ we have a Killing form $(\ ,\ )$ which is transferred to an inner product $\langle\ ,\ \rangle$ on the dual space V^* of V which can canonically be identified with the real vector space $X \otimes \mathbf{R}$. If r is a root in Φ, the coroot of G associated to r is defined to be the element h_r of Y such that $(h_r, h) = 2r(h)/\langle r, r\rangle$, for all $h \in Y$. In V there is an orthonormal basis $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$ such that the coroots in Y are the vectors $\pm\varepsilon_i \pm \varepsilon_j$, $1 \leqslant i, j \leqslant 4$. We fix the fundamental basis $\Delta = \{r_1, r_2, r_3, r_4\}$ in Φ for which the associated coroots are $h_1 = \varepsilon_1 - \varepsilon_2$, $h_2 = \varepsilon_2 - \varepsilon_3$, $h_3 = \varepsilon_3 - \varepsilon_4$, and $h_4 = \varepsilon_3 + \varepsilon_4$, respectively.

Let τ be the symmetry of the Dynkin diagram D_4 of G with nodes h_1, h_2, h_3, and h_4 such that τ: $h_1 \to h_3 \to h_4 \to h_1$ and $\tau(h_2) = h_2$. Then τ induces an isometry on V which again is denoted by τ. The triality automorphism $\sigma = \tau q$ of G is induced by τ times the field automorphism $z \to z^q$ of K. The simple group $^3D_4(q) = G_\sigma = \{g \in G \mid \sigma(g) = g\}$ is called the Steinberg-Tits triality. Its order $|G_\sigma| = q^{12}(q^8 + q^4 + 1)(q^6 - 1)(q^2 - 1)$.

The torus T is σ-stable. The restriction of $\sigma = q\tau$ onto T induces a linear transformation of V, again denoted by σ.

Let h: $\mathrm{Hom}(X, K^*) \to T$ be defined as follows. For every $\chi \in \mathrm{Hom}(X, K^*)$, $h(\chi) = t \in T$, where $\chi(\lambda) = \lambda(t)$ for all $\lambda \in X$. Then h is an isomorphism.

Let λ_1, λ_2, λ_3, and λ_4 be the fundamental weights in X. Each element $h(\chi) \in T$ can uniquely be written as

$$h(\chi) = \prod_{i=1}^{4} h(\chi_{h_i, z_i}),$$

where $\chi_{h_r, z}(\lambda) = z^{\lambda(h_r)}$ for $r \in \Phi$, $z \in K^*$, and where $\chi(\lambda_i) = z_i$ for $1 \leqslant i \leqslant 4$.

Let W be the Weyl group generated by all reflections w_r at the hyperplanes of V orthogonal to the coroots h_r, $r \in \Phi$. Then σ acts on W by $\sigma(w) = \sigma w \sigma^{-1} = \tau w \tau^{-1}$. In particular, $\sigma(w_r) = w_{\tau(r)}$. W acts also on T by $wh(\chi)$, where $(w\chi)(\lambda) = \chi(w^{-1}(\lambda))$ for all $\lambda \in X$. Furthermore, $w_{i \pm j}$ denotes the reflection at the hyperplane of V orthogonal to the coroot $\varepsilon_i \pm \varepsilon_j$.

Let r_0 be the highest root of Φ, and $\tilde{\Delta} = \Delta \cup \{-r_0\}$.

Let J be an arbitrary τ-invariant proper subset of $\tilde{\Delta}$, and W the Weyl group of the torus T. The normalizer of J in W is denoted by Ω_J. It is a σ-stable subgroup of W. Two elements w_1, $w_2 \in \Omega_J$ are called *σ-equivalent* if $w_1 = ww_2\sigma(w^{-1})$ for some $w \in \Omega_J$. The σ-equivalence class of $w \in \Omega_J$ is denoted by $[w]$, and $H^1(\sigma, \Omega_J)$ is the set of all σ-equivalence classes $[w]$ of Ω_J. The possibilities of J and Ω_J are given in Table 1.0, up to W-conjugacy.

TABLE 1.0

J	Ω_J
$J_0 = \{r_1, r_2, r_3, r_4\}$	$\Omega_{J_0} = 1$
$J_1 = \{r_1, r_3, r_4, -r_0\}$	$\Omega_{J_1} = \langle w_{1+4}w_{2+3}\rangle \times \langle w_{1-4}w_{1+4}\rangle \simeq (\mathbf{Z}_2)^2$
$J_2 = \{r_1, r_3, r_4\}$	$\Omega_{J_2} = \langle w_{1+2}\rangle \simeq \mathbf{Z}_2$
$J_3 = \{r_2, -r_0\}$	$\Omega_{J_3} = \langle w_{1-3}w_{2+4}w_{2-4}\rangle \simeq \mathbf{Z}_2$
$J_4 = \{-r_0\}$	$\Omega_{J_4} = \langle w_{1-2}\rangle \times \langle w_{3-4}\rangle \times \langle w_{3+4}\rangle \simeq (\mathbf{Z}_2)^3$
$J_5 = \varnothing$	$\Omega_{J_5} = W$

Let $\mathscr{C}_J$ be the collection of all σ-stable G-conjugates of $C_G(x)$ where x is a semisimple element of G with $r(x) = 1$ for all $r \in J$. Then the group G_σ acts on $\mathscr{C}_J$ by conjugation. If $J = \varnothing$ is the empty set, then $\Omega_\varnothing = W$, and x is a regular element of G_σ. There is a one-to-one correspondence between the G_σ-orbits of σ-stable maximal tori of G and the classes of $H^1(\sigma, W)$, see [**1**, p. 186]. It is known for the triality $G_\sigma = {}^3D_4(q)$ that $|H^1(\sigma, W)| = 7$; cf. [**14**].

Let T be a σ-stable maximal torus of G, with Weyl group $W = N_G(T)/T$. If T' is a σ-stable maximal torus of G, then there is a unique class $[w_j] \in H^1(\sigma, W)$ with $j \in \{0, 1, \dots, 6\}$ such that T'_σ is G-conjugate to $T_j = T_{w_j\sigma} = \{t \in T \mid w_j\sigma(t) = t\}$.

In particular, the element $h(\chi) = \prod_{i=1}^4 h(\chi_{h_i}, z_i) \in T$ belongs to T_j if and only if

$$h(\chi) = w_j\sigma h(\chi) = \prod_{i=1}^{4} h\left(\chi_{w_j\tau(h_i), z_i q}\right).$$

For the sake of simplicity, each element $h(\chi) = \prod_{i=1}^4 h(\chi_{h_i, z_i}) \in T$ is denoted by $h(\chi) = (z_1, z_2, z_3, z_4)$. With this notation we can parametrize all the elements of the tori T_j.

LEMMA 1.1. *Let $q \neq 2$. Let T' be a maximal σ-stable torus of G corresponding to the class $[w_j] \in H^1(\sigma, W)$, $j \in \{0, 1, \dots, 6\}$, and let $T_j = T_{w_j\sigma}$. Then the Weyl group W_j of T_j is given by*

$$W_j = C_{w,\sigma}(w_j) = \left\{w \in W \mid w w_j \sigma(w)^{-1} = w_j\right\} \cong N_{G_\sigma}(T_j)/T_j.$$

PROPOSITION 1.2 (P. C. GAGER). *The structure of the maximal tori T_j of G_σ and their Weyl groups W_j is given in Table* 1.1.

TABLE 1.1

$[w_j] \in H^1(\sigma, W)$	T_j	W_j
$w_0 = 1 \in W$	$T_0 = \{(z_1, z_2, z_1^q, z_1^{q^2}) \mid z_1^{q^3-1} = z_2^{q-1} = 1\}$ $T_0 \cong \mathbf{Z}_{q^3-1} \times \mathbf{Z}_{q-1}$	$W_\sigma \cong D_{12}$
$w_1 = w_{1+2}$	$T_1 = \{(z, z^{1-q^3}, z^{q^4}, z^{q^2}) \mid z^{(q^3-1)(q+1)} = 1\}$ $T_1 \cong \mathbf{Z}_{(q^3-1)(q+1)}$	$\mathbf{Z}_2 \times \mathbf{Z}_2$
$w_2 = -w_{r_0}$	$T_2 = \{(z, z^{q^3+1}, z^{q^4}, z^{q^2}) \mid z^{(q^3+1)(q-1)} = 1\}$ $T_2 \cong \mathbf{Z}_{(q^3+1)(q-1)}$	$\mathbf{Z}_2 \times \mathbf{Z}_2$
$w_3 = w_{1+2}w_{2-3}$	$T_3 = \{(z_1, z_2, z_1^q z_2, (z_1^{-1}z_2)^{q+1}) \mid z_i^{q^2+q+1} = 1\}$ $T_3 \cong \mathbf{Z}_{q^2+q+1} \times \mathbf{Z}_{q^2+q+1}$	$\mathrm{SL}_2(3)$
$w_4 = -w_{1+2}w_{2-3}$	$T_4 = \{(z_1, z_2, z_1^{-q}z_2, (z_1z_2^{-1})^{q-1}) \mid z_i^{q^2-q+1} = 1\}$ $T_4 \cong \mathbf{Z}_{q^2-q+1} \times \mathbf{Z}_{q^2-q+1}$	$\mathrm{SL}_2(3)$
$w_5 = w_{1-2}w_{2-3}$	$T_5 = \{(z, z^{q^3+1}, z^q, z^{q^2}) \mid z^{q^4-q^2+1} = 1\}$ $T_5 \cong \mathbf{Z}_{q^4-q^2+1}$	$\mathbf{Z}_4$
$w_6 = -1$	$T_6 = \{(z_1, z_2, z_1^{-q}, z_1^{q^2}) \mid z_1^{q^3+1} = z_2^{q+1} = 1\}$ $T_6 \cong \mathbf{Z}_{q^3+1} \times \mathbf{Z}_{q+1}$	$W_\sigma \cong D_{12}$

2. Structure of the centralizers of the semisimple elements and the determination of the conjugacy classes. For each $i \in \{0, 1, \dots, 5\}$ let E_i denote the Dynkin diagram type of the root system Φ_{J_i} generated by J_i. Let $\mathscr{C}_{J_i}$ be the collection of all σ-stable G-conjugates of $C_G(x)$, where x is an element of the maximal torus T of G. By

Corollary 3 of [3] there is a one-to-one correspondence between the G_σ-orbits of $\mathscr{C}_{J_i}$ and the classes of $H^1(\sigma, \Omega_{J_i})$. Therefore each G_σ-orbit of $\mathscr{C}_{J_i}$ can be parametrized by a pair $(E_i, [w])$, where $[w] \in H^1(\sigma, \Omega_{J_i})$.

Proposition 2.1. *Let s_i be a representative of a semisimple conjugacy class $s_i^{G_\sigma}$ of G_σ whose centralizer $C_G(s_i)$ is in the orbit parametrized by the pair $(E_i, [w])$. Then the semisimple conjugacy classes are classified in Table 2.1.*

Table 2.1

$(E_i,[w])$	s_i, q even	s_i, q odd
$(E_0,[1])$	$s_1 = (1,1,1,1)$	$s_1 = (1,1,1,1)$
$(E_1,[1])$	s_2	$s_2 = (t,1,t,t)$, $t^2 = 1,\ t \neq 1$
$(E_2,[1])$	$s_3 = (t,t^2,t,t)$, $t^{q-1} = 1,\ t \neq 1$	$s_3 = (t,t^2,t,t)$, $t^{q-1} = 1,\ t^2 \neq 1$
$(E_3,[1])$	$s_4 = (t,1,t^q,t^{q^2})$, $t^{q^2+q+1} = 1,\ t \neq 1$	$s_4 = (t,1,t^q,t^{q^2})$, $t^{q^2+q+1} = 1, t \neq 1$
$(E_4,[1])$	$s_5 = (t,1,t^q,t^{q^2})$, $t^{q^3-1} = 1,\ t^{q^2+q+1} \neq 1$	$s_5 = (t,1,t^q,t^{q^2})$, $t^{q^3-1} = 1,\ t^{q^2+q+1} \neq 1,\ t^2 \neq 1$
$(E_5,[1])$	$s_6 = (t_1,t_2,t_1^q,t_1^{q^2})$, $t_1^{q^3-1} = t_2^{q-1} = 1,\ t_2 \neq 1,\ t_1^2 \neq t_2$, $t_1^{q^2+q+1} \neq t_2,\ t_1^{q^2+q+1} \neq t_2^2$	$s_6(t_1,t_2,t_1^q,t_1^{q^2})$, $t_1^{q^3-1} = t_2^{q-1} = 1,\ t_2 \neq 1,\ t_1^2 \neq t_2$, $t_1^{q^2+q+1} \neq t_2,\ t_1^{q^2+q+1} \neq t_2^2$
$(E_2,[w_{1+2}])$	$s_7 = (t,t^2,t,t)$, $t \neq 1,\ t^{q+1} = 1$	$s_7 = (t,t^2,t,t)$, $t^2 \neq 1,\ t^{q+1} = 1$
$(E_5,[w_{1+2}])$	$s_8 = (t,t^{1-q^3},t^{q^4},t^{q^2})$, $t^{(q^3-1)(q+1)} = 1,\ t^{q^3-1} \neq 1 \neq t^{q+1}$	$s_8 = (t,t^{1-q^3},t^{q^4},t^{q^2})$, $t^{(q^3-1)(q+1)} = 1,\ t^{q^3-1} \neq 1 \neq t^{q+1}$
$(E_3,[-w_{1+2}])$	$s_9 = (t,1,t^{-q},t^{q-1})$, $t^{q^2-q+1} = 1,\ t \neq 1$	$s_9 = (t,1,t^{-q},t^{q-1})$, $t^{q^2-q+1} = 1,\ t^2 \neq 1$
$(E_4,[-w_{1+2}])$	$s_{10} = (t,1,t^{-q},t^{q^2})$, $t^{q^3+1} = 1,\ t^{q^2-q+1} \neq 1,\ t \neq 1$	$s_{10} = (t,1,t^{-q},t^{q^2})$, $t^{q^3+1} = 1,\ t^{q^2-q+1} \neq 1,\ t^2 \neq 1$
$(E_5,[-w_{1+2}])$	$s_{11} = (t,t^{q^3+1},t^{q^4},t^{q^2})$, $t^{(q^3+1)(q-1)} = 1,\ t^{q-1} \neq 1,\ t^{q^3+1} \neq 1$	$s_{11} = (t,t^{q^3+1},t^{q^4},t^{q^2})$, $t^{(q^3+1)(q-1)} = 1,\ t^{q^3+1} \neq 1,\ t^{q-1} \neq 1$
$(E_5,[w_{1+2}w_{2-3}])$	$s_{12} = (t_1,t_2,t_1^q t_2,(t_1^{-1}t_2)^{q+1})$, $t_1^{q^2+q+1} = t_2^{q^2+q+1} = 1,\ t_1 \neq t_2$	$s_{12} = (t_1,t_2,t_1^q t_2,(t_1^{-1}t_2)^{q+1})$, $t_1^{q^2+q+1} = t_2^{q^2+q+1} = 1,\ t_1 \neq t_2$
$(E_5,[-w_{1+2}w_{2-3}])$	$s_{13} = (t_1,t_2,t_1^{-q}t_2,(t_1t_2^{-1})^{q-1})$, $t_1^{q^2-q+1} = t_2^{q^2-q+1} = 1,\ t_1 \neq t_2$	$s_{13} = (t_1,t_2,t_1^{-q}t_2,(t_1t_2^{-1})^{q-1})$, $t_1^{q^2-q+1} = t_2^{q^2-q+1} = 1,\ t_1 \neq t_2$
$(E_5,[w_{1-2}w_{2-3}])$	$s_{14} = (t,t^{q^3+1},t^q,t^{q^2})$, $t^{q^4-q^2+1} = 1,\ t \neq 1$	$s_{14} = (t,t^{q^3+1},t^q,t^{q^2})$, $t^{q^4-q^2+1} = 1,\ t \neq 1$
$(E_5,[-1])$	$s_{15} = (t_1,t_2,t_1^{-q},t^{q^2})$, $t_1^{q^3+1} = t_2^{q+1} = 1,\ t_1^2 \neq t_2 \neq 1,\ t_1 \neq t_2$, $t_1^{q^2-q+1} \neq t_2^2,\ t_1^{q^2-q+1} \neq t_2$	$s_{15} = (t_1,t_2,t_1^{-q},t_1^{q^2})$, $t_1^{q^3+1} = t_2^{q+1} = 1,\ t_1^2 \neq 1 \neq t_2^2,\ t_1 \neq t_2$, $t_1^{q^2-q+1} \neq t_2^2,\ t_1^{q^2-q+1} \neq t_2$

Let $x \in G_\sigma$ be semisimple contained in the maximal torus T of G. By Proposition 2.3.2 of [4] there is a proper subset J of Δ such that $C_G(x)$ is generated by T and the root subgroups X_r, $r \in J$.

Let $M = \{X_r \mid r \in J\}$ and let S be the connected component of the center of $C_G(x)$. Then M is semisimple, S is a torus, $C_G(x) = MS$ and $M \cap S$ is finite. Moreover, the order

$$|C_{G_\sigma}(x)| = |M_\sigma| \cdot |S_\sigma|.$$

Furthermore, $M_{\sigma u}$ denotes the subgroup of M_σ generated by all its unipotent elements. Certainly $M_{\sigma u}$ is a characteristic subgroup of $C_{G_\sigma}(x)$.

PROPOSITION 2.2. *Let $s_i \neq 1$ be a representative of a nonregular semisimple conjugacy class of G_σ. The structure of its centralizer $C = C_{G_\sigma}(s_i)$ is as given in Tables* 2.2a *and* 2.2b.

In particular, $M_\sigma = M_{\sigma u}$ for every $s_i \neq s_2$.

TABLE 2.2a. Even q

class	$M_{\sigma u}$	S_σ	$\lvert C : M_{\sigma u} * S_\sigma \rvert$	C, C', or C/S_σ
s_3	$SL_2(q^3)$	$\mathbf{Z}_{q-1}$	1	$C \simeq SL_2(q^3) \times \mathbf{Z}_{q-1}$
s_4 if $3 \nmid q-1$	$SL_3(q)$	$\mathbf{Z}_{q^2+q+1}$	1	$C \simeq SL_3(q) \times \mathbf{Z}_{q^2+q+1}$
s_4 if $3 \mid q-1$	$SL_3(q)$	$\mathbf{Z}_{q^2+q+1}$	3	$C/S_\sigma \simeq PGL_3(q)$
s_5	$SL_2(q)$	$\mathbf{Z}_{q^3-1}$	1	$C \simeq SL_2(q) \times \mathbf{Z}_{q^3-1}$
s_7	$SL_2(q^3)$	$\mathbf{Z}_{q+1}$	1	$C \simeq SL_2(q^3) \times \mathbf{Z}_{q+1}$
s_9 if $3 \nmid q+1$	$SU_3(q)$	$\mathbf{Z}_{q^2-q+1}$	1	$C \simeq SU_3(q) \times \mathbf{Z}_{q^2-q+1}$
s_9 if $3 \mid q+1$	$SU_3(q)$	$\mathbf{Z}_{q^2-q+1}$	3	$C/S_\sigma \simeq PU_3(q)$
s_{10}	$SL_2(q)$	$\mathbf{Z}_{q^3+1}$	1	$C \simeq SL_2(q) \times \mathbf{Z}_{q^3+1}$

TABLE 2.2b. Odd q

class	$M_{\sigma u}$	S_σ	$\lvert C : M_{\sigma u} * S_u \rvert$	C, C', or C/S_σ
s_2	$SL_2(q^3) * SL_2(q)$	1	2	$C' = SL_2(q^3) * SL_2(q)$
s_3	$SL_2(q^3)$	$\mathbf{Z}_{q-1}$	2	$C/S_\sigma \simeq PGL_2(q^3)$
s_4 if $3 \nmid q-1$	$SL_3(q)$	$\mathbf{Z}_{q^2+q+1}$	1	$C \simeq SL_3(q) \times \mathbf{Z}_{q^2+q+1}$
s_4 if $3 \mid q-1$	$SL_3(q)$	$\mathbf{Z}_{q^2+q+1}$	3	$C/S_\sigma \simeq PGL_3(q)$
s_5	$SL_2(q)$	$\mathbf{Z}_{q^3-1}$	2	$C/S_\sigma \simeq PGL_2(q)$
s_7	$SL_2(q^3)$	$\mathbf{Z}_{q+1}$	2	$C/S_\sigma \simeq PGL_2(q^3)$
s_9 if $3 \nmid q+1$	$SU_3(q)$	$\mathbf{Z}_{q^2-q+1}$	1	$C \simeq SU_3(q) \times \mathbf{Z}_{q^2-q+1}$
s_9 if $3 \mid q+1$	$SU_3(q)$	$\mathbf{Z}_{q^2-q+1}$	3	$C/S_\sigma \simeq PU_3(q)$
s_{10}	$SL_2(q)$	$\mathbf{Z}_{q^3+1}$	2	$C/S_\sigma \simeq PGL_2(q)$

PROOF. In Table 7 of Deriziotis [4, p. 140], for each centralizer $C_{G_\sigma}(s_i)$ the isogeny class of the groups M_σ and the orders of the cyclic groups S_σ are given. Using similar methods as in Iwahori's paper [1, p. 281], the precise group structure of $C_{G_\sigma}(s_i)$ can be determined.

In order to find the mixed conjugacy classes, we use Spaltenstein's [14] results on the orders of the centralizers $C_{G_\sigma}(u_j)$ of the unipotent elements u_j of G_σ, with the following notation. (See Table A.)

Table A

unipotent class	1	u_1	u_2	u_3	u_4	u_5	u_6	u_7
notation of [14] for even q	$\varnothing$	A_1	$3A_1$	A_2'	A_2''	$D_4(a_1)$	D_4'	D_4''
for odd q	$\varnothing$	A_1	$3A_1'$	A_2'	A_2''	$D_4(a_1)$	D_4	

PROPOSITION 2.3. *G_σ has q^3+q^2+q and q^3+q^2+q-2 mixed conjugacy classes with representatives $s_i \cdot u_j = u_j \cdot s_i$ for odd and even q, respectively, where $s_i \neq 1$ is a representative of a nonregular semisimple and $u_j \neq 1$ is a representative of a unipotent conjugacy class of G_σ. These mixed conjugacy classes are given in Table* 2.4.

Furthermore, if $k(G_\sigma)$ denotes the number of all conjugacy classes of G_σ, then

$$k(G_\sigma) = \begin{cases} q^4+q^3+q^2+q+6, & \text{if } q \text{ is odd},\\ q^4+q^3+q^2+q+5, & \text{if } q \text{ is even}. \end{cases}$$

TABLE 2.4

ss class	unipotent classes of $C_{G_\sigma}(s_i)$	Number of mixed classes $(s_iu_j)G_\sigma$ q odd	q even
s_2	u_1, u_2, u_3, u_4	4	—
s_3	u_2	$\frac{1}{2}(q-3)$	$\frac{1}{2}(q-2)$
s_4	u_1, u_3	q^2+q	q^2+q
s_5	u_1	$\frac{1}{2}(q^3-q^2-q-3)$	$\frac{1}{2}(q^3-q^2-q-2)$
s_7	u_2	$\frac{1}{2}(q-1)$	$\frac{1}{2}q$
s_9	u_1, u_4	q^2-q	q^2-q
s_{10}	u_1	$\frac{1}{2}(q^3-q^2+q-1)$	$\frac{1}{2}(q^3-q^2+q)$

3. Deligne-Lusztig characters. In this section we determine the values of the Deligne-Lusztig characters of $G_\sigma = {}^3D_4(q)$. Concerning the definition and the main properties of these class functions we refer to Carter [2] and Lusztig [11].

Let T_0 be a maximally split torus of the connected reductive group G, X its character group, and $V = X \otimes \mathbf{R}$. Then $\sigma = q\tau$ acts on V. *The relative rank* rel rank G *of* G is the number of eigenvalues of σ on V which are equal to q; see Carter [2].

DEFINITION. $\varepsilon_G = (-1)^{\text{rel rank } G}$.

By Corollary 6.5.7 of [2], $\varepsilon_G = \varepsilon_{T_0} = 1$ in our case $G_\sigma = {}^3D_4(q)$.

LEMMA 3.1. *Let $s \neq 1$ be a semisimple element of $G_\sigma = {}^3D_4(q)$. Then its centralizer $C_G(s)$ has sign $\varepsilon_{C_G(s)}$ which is given by Table* B.

Table B

class s	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}	s_{11}	s_{12}	s_{13}	s_{14}	s_{15}
$\varepsilon_{C_G(s)}$	1	1	1	1	1	-1	-1	-1	-1	-1	1	1	1	1

PROOF. This follows easily from Proposition 2.2 and Corollary 6.5.7 of [**2**].

Let s be a semisimple element and T a maximal torus of G_σ. Then as in group theory we write $s \in_{G_\sigma} T$, if $s^g \in T$ for some $g \in G_\sigma$. In the following, $C(s)$ denotes the centralizer $C_{G_\sigma}(s)$. If Θ is a linear character of T, then $R_{T,\Theta}$ is the corresponding Deligne-Lusztig character of G_σ. For any unipotent element $u \in G_\sigma$ the Green function Q_T has value $Q_T(u) = R_{T,1}(u)$. If it is necessary to indicate the ambient group we also write $Q_T^G(u)$ and $R_{T,\Theta}^G$.

For the sake of completeness the following subsidiary result is given.

LEMMA 3.2. *Let T be a maximal torus of $G_\sigma = {}^3D_4(q)$ with Weyl group W_T. Then for every linear character Θ of T and every $x = su \in G_\sigma$ in Jordan form the Deligne-Lusztig character $R_{T,\Theta}^G$ has value*

$$R_{T,\Theta}^G(x) = \begin{cases} \dfrac{\varepsilon_{C(s)}\varepsilon_T |C(s)|_{p'}}{|T|}\tilde{\Theta}(s) & \text{if } u = 1 \text{ and } s \in_{G_\sigma} T, \\ \tilde{\Theta}(s) Q_T^{C(s)}(u) & \text{if } u \neq 1 \text{ and } s \in_{G_\sigma} T, \\ 0 & \text{if } s \notin_{G_\sigma} T, \end{cases}$$

where

$$\tilde{\Theta}(s) = \frac{1}{|C_{W_T}(s)|} \sum_{w \in W_T} \Theta(wsw^{-1}).$$

With the notation of Propositions 2.2 and 2.3 we state

LEMMA 3.3. *Let $s \neq 1$ be a semisimple element of $G_\sigma = {}^3D_4(q)$ and u a unipotent element of $C(s)$. Let T be a maximal torus of G_σ contained in $C(s)$. Then the values $Q_T(u)$ of the Green functions of $C(s)$ are given by*

(a) $Q_T(u) = 1$, *if $s \in (s_i)^{G_\sigma}$ and $i \in \{3, 5, 7, 10\}$.*

(b)

$C(s_2)$	u_1	u_2	u_3	u_4
Q_{T_0}	$q+1$	q^3+1	1	1
Q_{T_1}	$1-q$	q^3+1	1	1
Q_{T_2}	$q+1$	$1-q^3$	1	1
Q_{T_6}	$1-q$	$1-q^3$	1	1

(c)

$C(s_4)$	u_1	u_3	$C(s_9)$	u_1	u_4
Q_{T_0}	$1+2q$	1	Q_{T_2}	1	1
Q_{T_1}	1	1	Q_{T_4}	$q+1$	1
Q_{T_3}	$1-q$	1	Q_{T_6}	$1-2q$	1

Proof. Every Green function Q_T is a linear combination of the unipotent characters of $C(s)$. Using then the character tables of $\mathrm{SL}_2(q)$, $\mathrm{SL}_3(q)$, $\mathrm{SU}_3(q)$ of [**6** and **13**] it is easy to compute the given values of $Q_T(u)$, because Proposition 2.2 gives the group structure of $C(s)$.

With the notation of Propositions 1.2 and 2.1 we state the following result.

Lemma 3.4. *Let $q \neq 2$ and s be a semisimple element. Then for $0 \leqslant j \leqslant 6$, the centralizer $C_{W_j}(s)$ of s in the Weyl group W_j is as given in Table* 3.4.

Table 3.4

s	$C_{W(T_0)}(s)$	$C_{W(T_1)}(s)$	$C_{W(T_2)}(s)$	$C_{W(T_3)}(s)$	$C_{W(T_4)}(s)$	$C_{W(T_5)}(s)$	$C_{W(T_6)}(s)$
1	$W(G_2) \simeq D_{12}$	$\mathbf{Z}_2 \times \mathbf{Z}_2$	$\mathbf{Z}_2 \times \mathbf{Z}_2$	$Q_8 \cdot \mathbf{Z}_3$	$Q_8 \cdot \mathbf{Z}_3$	$\mathbf{Z}_4$	$W(G_2) \simeq D_{12}$
s_2	$\mathbf{Z}_2 \times \mathbf{Z}_2$	$\mathbf{Z}_2 \times \mathbf{Z}_2$	$\mathbf{Z}_2 \times \mathbf{Z}_2$				$\mathbf{Z}_2 \times \mathbf{Z}_2$
s_3	$\mathbf{Z}_2$		$\mathbf{Z}_2$				
s_4	S_3	$\mathbf{Z}_2$		$\mathbf{Z}_3$			
s_5	$\mathbf{Z}_2$	$\mathbf{Z}_2$					
s_6	1						
s_7		$\mathbf{Z}_2$					$\mathbf{Z}_2$
s_8		1					
s_9			$\mathbf{Z}_2$		$\mathbf{Z}_3$		S_3
s_{10}			$\mathbf{Z}_2$				$\mathbf{Z}_2$
s_{11}			1				
s_{12}				1			
s_{13}					1		
s_{14}						1	
s_{15}							1

In the following subsidiary result $H \rtimes U$ denotes the semidirect product of the normal subgroup H with the subgroup U of the finite group X.

Lemma 3.5. *Let $q = 2$. Let s be a semisimple element and $W_j = N_{G_\sigma}(T_j)/T_j$, $0 \leqslant j \leqslant 6$. Then the centralizer $C_{W_j}(s)$ of s in W_j is as given in Table* 3.5.

Table 3.5

s	$C_{W(T_0)}(s)$	$C_{W(T_1)}(s)$	$C_{W(T_2)}(s)$	$C_{W(T_3)}(s)$	$C_{W(T_4)}(s)$	$C_{W(T_5)}(s)$	$C_{W(T_6)}(s)$
1	$\mathrm{SL}_3(2) \rtimes \mathbf{Z}_2$	$\mathbf{Z}_2 \times \mathbf{Z}_2$	$\mathrm{SL}_2(2) \rtimes \mathbf{Z}_2$	$Q_8 \cdot \mathbf{Z}_3$	$Q_8 \cdot \mathbf{Z}_3$	$\mathbf{Z}_4$	$W(G_2) \simeq D_{12}$
s_4	$\mathrm{SL}_3(2)$	$\mathbf{Z}_2$		$\mathbf{Z}_3$			
s_6	1						
s_7		$\mathbf{Z}_2$					$\mathbf{Z}_2$
s_8		1					
s_9			S_3		$\mathbf{Z}_3$		S_3
s_{10}			S_3				$\mathbf{Z}_2$
s_{12}				1			
s_{14}						1	

Proof. Since no root vanishes on the tori T_1, T_3, T_4, T_5, and T_6 the proof of Lemma 3.4 remains valid for these cases by Veldkamp's theorem [**15**].

By Proposition 2.2 $H = C_{G_\sigma}(T_0) \simeq \mathrm{SL}_3(2) \times \mathbf{Z}_7$. Propositions 1.2 and 2.1 imply that $N_{G_\sigma}(T_0)/H \simeq \mathbf{Z}_2$. Thus $N_{G_\sigma}(T_0) \simeq \mathrm{SL}_3(2) \rtimes \mathbf{Z}_2$. The remaining cases are proved similarly.

In order to give the values of the Deligne-Lusztig characters $R_{T,\Theta}$ we introduce the following

Notation. For a σ-stable maximal torus T of G we fix an isomorphism $T_\sigma \simeq \hat{T}_\sigma = \mathrm{Hom}(T_\sigma, \mathbb{C}^*)$. The linear character of T_σ corresponding to $s \in T_\sigma$ under this isomorphism will be denoted by $\hat{s}$.

Let T_j be a maximal torus of G_σ, $0 \leqslant j \leqslant 6$, and $s_i \in T_j$ be a representative of a semisimple conjugacy class of G_σ, where $i \in \{2, 3, \ldots, 15\}$.

TABLE 3.6. Deligne-Lusztig characters

	s_2	s_2u_1	s_2u_2	s_2u_3	s_2u_4
$R_{0,i}$ $i = 2,3,4,5,6$	$(q^3+1)(q+1)\eta_{0,i}(s_2)$	$(q+1)\eta_{0,i}(s_2)$	$(q^3+1)\eta_{0,i}(s_2)$	$\eta_{0,i}(s_2)$	$\eta_{0,i}(s_2)$
$R_{1,i}$ $i = 2,4,5,7,8$	$-(q^3+1)(q-1)\eta_{1,i}(s_2)$	$-(q-1)\eta_{1,i}(s_2)$	$(q^3-1)\eta_{1,i}(s_2)$	$\eta_{1,i}(s_2)$	$\eta_{1,i}(s_2)$
$R_{2,i}$ $i = 2,3,9,10,11$	$-(q^3-1)(q+1)\eta_{2,i}(s_2)$	$(q+1)\eta_{2,i}(s_2)$	$-(q^3-1)\eta_{2,i}(s_2)$	$\eta_{2,i}(s_2)$	$\eta_{2,i}(s_2)$
$R_{3,i}$ $i = 4,12$	0	0	0	0	0
$R_{4,i}$ $i = 9,13$	0	0	0	0	0
$R_{5,i}$ $i = 14$	0	0	0	0	0
$R_{6,i}$ $i = 2,7,9,10,15$	$(q^3-1)(q-1)\eta_{6,i}(s_2)$	$-(q-1)\eta_{6,i}(s_2)$	$-(q^3-1)\eta_{6,i}(s_2)$	$\eta_{6,i}(s_2)$	$\eta_{6,i}(s_2)$

	s_3	s_3u_2	s_4	s_4u_1	s_4u_3
$R_{0,i}$ $i = 2,3,4,5,6$	$(q^3+1)\eta_{0,i}(s_3)$	$\eta_{0,i}(s_3)$	$(q+1)(q^2+q+1)\eta_{0,i}(s_4)$	$(2q+1)\eta_{0,i}(s_4)$	$\eta_{0,i}(s_4)$
$R_{1,i}$ $i = 2,4,5,7,8$	0	0	$-(q^3-1)\eta_{1,i}(s_4)$	$\eta_{1,i}(s_4)$	$\eta_{1,i}(s_4)$
$R_{2,i}$ $i = 2,3,9,10,11$	$-(q^3-1)\eta_{2,i}(s_3)$	$\eta_{2,i}(s_3)$	0	0	0
$R_{3,i}$ $i = 4,12$	0	0	$(q-1)(q^2-1)\eta_{3,i}(s_4)$	$-(q-1)\eta_{3,i}(s_4)$	$\eta_{3,i}(s_4)$
$R_{4,i}$ $i = 9,13$	0	0	0	0	0
$R_{5,i}$ $i = 14$	0	0	0	0	0
$R_{6,i}$ $i = 2,7,9,10,15$	0	0	0	0	0

	s_5	s_5u_1	s_6	s_7	s_7u_2	s_8
$R_{0,i}$ $i = 2,3,4,5,6$	$(q+1)\eta_{0,i}(s_5)$	$\eta_{0,i}(s_5)$	$\eta_{0,i}(s_6)$	0	0	0
$R_{1,i}$ $i = 2,4,5,7,8$	$-(q-1)\eta_{1,i}(s_5)$	$\eta_{1,i}(s_5)$	0	$(q^3+1)\eta_{1,i}(s_7)$	$\eta_{1,i}(s_7)$	$\eta_{1,i}(s_8)$
$R_{2,i}$ $i = 2,3,9,10,11$	0	0	0	0	0	0
$R_{3,i}$ $i = 4,12$	0	0	0	0	0	0
$R_{4,i}$ $i = 9,13$	0	0	0	0	0	0
$R_{5,i}$ $i = 14$	0	0	0	0	0	0
$R_{6,i}$ $i = 2,7,9,10,15$	0	0	0	$-(q^3-1)\eta_{6,i}(s_7)$	$\eta_{6,i}(s_7)$	0

TABLE 3.6 (continued)

	s_9	s_9u_1	s_9u_4	s_{10}	$s_{10}u_1$
$R_{0,i}$ $i = 2,3,4,5,6$	0	0	0	0	0
$R_{1,i}$ $i = 2,4,5,7,8$	0	0	0	0	0
$R_{2,i}$ $i = 2,3,9,10,11$	$(q^3+1)\eta_{2,i}(s_9)$	$\eta_{2,i}(s_9)$	$\eta_{2,i}(s_9)$	$(q+1)\eta_{2,i}(s_{10})$	$\eta_{2,i}(s_{10})$
$R_{3,i}$ $i = 4,12$	0	0	0	0	0
$R_{4,i}$ $i = 9,13$	$-(q+1)(q^2-1)\eta_{4,i}(s_9)$	$(q+1)\eta_{4,i}(s_9)$	$\eta_{4,i}(s_9)$	0	0
$R_{5,i}$ $i = 14$	0	0	0	0	0
$R_{6,i}$ $i = 2,7,9,10,15$	$-(q-1)(q^2-q+1)\eta_{6,i}(s_9)$	$-(2q-1)\eta_{6,i}(s_9)$	$\eta_{6,i}(s_9)$	$-(q-1)\eta_{6,i}(s_{10})$	$\eta_{6,i}(s_{10})$

	s_{11}	s_{12}	s_{13}	s_{14}	s_{15}
$R_{0,i}$ $i = 2,3,4,5,6$	0	0	0	0	0
$R_{1,i}$ $i = 2,4,5,7,8$	0	0	0	0	0
$R_{2,i}$ $i = 2,3,9,10,11$	$\eta_{2,i}(s_{11})$	0	0	0	0
$R_{3,i}$ $i = 4,12$	0	$\eta_{3,i}(s_{12})$	0	0	0
$R_{4,i}$ $i = 9,13$	0	0	$\eta_{4,i}(s_{13})$	0	0
$R_{5,i}$ $i = 14$	0	0	0	$\eta_{5,i}(s_{14})$	0
$R_{6,i}$ $i = 2,7,9,10,15$	0	0	0	0	$\eta_{6,i}(s_{15})$

We set $R_{j,i} = R_{T_j,\hat{s}_i}$ and

$$\mathcal{N}_{j,i}(s) = \frac{1}{\left|C_{W_j(s)}\right|} \sum_{w \in W_j} \hat{s}_i(s^w),$$

where W_j is the Weyl group of T_j.

4. The irreducible characters. By Spaltenstein's paper [**14**] the character values of the unipotent irreducible characters of the triality groups $^3D_4(q)$ are known. In this section we determine the remaining irreducible characters of G_σ by means of the Deligne-Lusztig theory; see Lusztig's book [**11**].

G_σ is isomorphic to the fixed points $G^*_{\sigma^*}$ of the triality endomorphism σ^* of the adjoint group G^* of type D_4, which is dual to G; see [**2**, p. 112]. As G^* has a connected center, Lusztig has shown in [**11**] that there is a bijective map $\chi \to (\chi_s, \chi_u)$ between the irreducible characters χ of G_σ and pairs (χ_s, χ_u), where χ_s is a semisimple character of G_σ and χ_u is a unipotent character of the centralizer $C(s)$ of a semisimple element $s \in G$. Furthermore, this bijection satisfies the following conditions:

$$\chi(1) = \chi_s(1)\chi_u(1), \tag{4.1}$$

$$(\chi, \varepsilon_T R_{T,\theta})_{G_\sigma} = (\chi_u, \varepsilon_{C(s)}\varepsilon_T R_{T,1})_{C(s)}, \tag{4.2}$$

because $\varepsilon_G = 1$ by Corollary 6.5.7 of [**2**].

The unipotent irreducible characters of $\mathrm{SL}_2(q^i)$ are the trivial character 1 and the Steinberg character St. Following the notation of Simpson's and Frame's [**13**] character tables, besides 1 and St the groups $\mathrm{SL}_3(q)$ and $\mathrm{SU}_3(q)$ each have another unipotent irreducible character denoted by χ_{qs} and $\chi_{qs'}$, respectively, where $s = q + 1$ and $s' = q - 1$.

Using now the notation of Proposition 2.1 for the semisimple conjugacy classes $s_i \neq 1$, $2 \leqslant i \leqslant 15$, and the structure of the centralizer $C(s_i)$ given in Proposition 2.2, every irreducible character $\chi = (\chi_s, \chi_u)$ of $G_\sigma = {}^3D_4(q)$ which is not unipotent can (up to conjugation) be uniquely denoted by

$$\chi = \begin{cases} \chi_i, & \text{if } \chi = (\chi_i, \varnothing), \text{ and } s_i \text{ is regular} \\ \chi_{i,1}, & \text{if } \chi = (\chi_{s_i}, 1), \text{ and } s_i \neq 1 \text{ is not regular} \\ \chi_{i,\mathrm{St}}, & \text{if } \chi = (\chi_{s_i}, \mathrm{St}) \\ \chi_{i,qs}, & \text{if } \chi = (\chi_{s_i}, \chi_{qs}) \\ \chi_{i,qs'}, & \text{if } \chi = (\chi_{s_i}, \chi_{qs'}) \\ \chi_{i,\mathrm{St}\,\mathrm{St}'}, & \text{if } i = 2 \text{ and St, St}' \text{ denote the} \\ & \text{Steinberg characters of } \mathrm{SL}_2(q^3), \\ & \mathrm{SL}_2(q), \text{ respectively.} \end{cases}$$

We keep Spaltenstein's [**14**] notation of the unipotent irreducible characters of G_σ. Their values are given in [**14**].

Therefore the following result and the table of the Deligne-Lusztig characters complete the character table of G_σ.

THEOREM 4.3. *With the values of the Deligne-Lusztig characters $R_{i,j}$ given in Table 3.6, the values of the nonunipotent irreducible characters χ of G_σ are determined as follows.*

(a)
$$\begin{aligned} \chi_{2,1} &= \tfrac{1}{4}(R_{0,2} + R_{2,2} + R_{1,2} + R_{6,2}) \\ \chi_{2,\mathrm{St}} &= \tfrac{1}{4}(R_{0,2} - R_{2,2} + R_{1,2} - R_{6,2}) \\ \chi_{2,\mathrm{St}'} &= \tfrac{1}{4}(R_{0,2} + R_{2,2} - R_{1,2} - R_{6,2}) \\ \chi_{2,\mathrm{St}\,\mathrm{St}'} &= \tfrac{1}{4}(R_{0,2} - R_{2,2} - R_{1,2} + R_{6,2}) \end{aligned}$$

(b)
$$\begin{aligned} \chi_{3,1} &= \tfrac{1}{2}(R_{0,3} + R_{2,3}) \\ \chi_{3,\mathrm{St}} &= \tfrac{1}{2}(R_{0,3} - R_{2,3}) \end{aligned}$$

(c)
$$\chi_{4,1} = \tfrac{1}{6}(R_{0,4} + 3R_{1,4} + 2R_{3,4})$$
$$\chi_{4,\mathrm{St}} = \tfrac{1}{6}(R_{0,4} - 3R_{1,4} + 2R_{3,4})$$
$$\chi_{4,qs} = \tfrac{1}{3}(R_{0,4} - R_{3,4})$$

(d)
$$\chi_{5,1} = \tfrac{1}{2}(R_{0,5} + R_{1,5})$$
$$\chi_{5,\mathrm{St}} = \tfrac{1}{2}(R_{0,5} - R_{1,5})$$

(e)
$$\chi_6 = R_{0,6}$$

(f)
$$\chi_{7,1} = -\tfrac{1}{2}(R_{1,7} + R_{6,7})$$
$$\chi_{7,\mathrm{St}} = -\tfrac{1}{2}(R_{1,7} - R_{6,7})$$

(g)
$$\chi_8 = -R_{1,8}$$

(h)
$$\chi_{9,1} = -\tfrac{1}{6}(R_{6,9} + 3R_{2,9} + 2R_{4,9})$$
$$\chi_{9,\mathrm{St}} = \tfrac{1}{6}(R_{6,9} - 3R_{2,9} + 2R_{4,9})$$
$$\chi_{9,qs'} = -\tfrac{1}{3}(R_{6,9} - R_{4,9})$$

(i)
$$\chi_{10,1} = -\tfrac{1}{2}(R_{2,10} + R_{6,10})$$
$$\chi_{10,\mathrm{St}} = -\tfrac{1}{2}(R_{2,10} - R_{6,10})$$

(j)
$$\chi_{11} = -R_{2,11}$$

(k)
$$\chi_{12} = R_{3,12}$$

(l)
$$\chi_{13} = R_{4,13}$$

(m)
$$\chi_{14} = R_{5,14}$$

(n)
$$\chi_{15} = R_{6,15}$$

PROOF. Let $C(s_2)$ be the centralizer of the unique involution $s_2 \neq 1$ in case q is odd. Its commutator subgroup $C(s_2)' = \mathrm{SL}_2(q^3) * \mathrm{SL}_2(q)$ by Proposition 2.2. Since the central involution s_2 of $C(s_2)$ is in the kernel of each unipotent irreducible character of $C(s_2)'$ the Green functions of $C(s_2)$ are given by

$$\begin{aligned} R_{T_0,1}^{C(s_2)} &= (1 + \mathrm{St}) \otimes (1 + \mathrm{St}') \\ &= 1 \otimes 1 + \mathrm{St} \otimes 1 + 1 \otimes \mathrm{St}' + \mathrm{St} \otimes \mathrm{St}', \\ R_{T_1,1}^{C(s_2)} &= (1 + \mathrm{St}) \otimes (1 - \mathrm{St}') \\ &= 1 \otimes 1 + \mathrm{St} \otimes 1 - 1 \otimes \mathrm{St}' - \mathrm{St} \otimes \mathrm{St}', \end{aligned}$$

$$R^{C(s_2)}_{T_2,1} = (1 - \mathrm{St}) \otimes (1 + \mathrm{St}')$$
$$= 1 \otimes 1 - \mathrm{St} \otimes 1 + 1 \otimes \mathrm{St}' - \mathrm{St} \otimes \mathrm{St}',$$
$$R^{C(s_2)}_{T_6,1} = (1 - \mathrm{St}) \otimes (1 - \mathrm{St}')$$
$$= 1 \otimes 1 - \mathrm{St} \otimes 1 - 1 \otimes \mathrm{St}' + \mathrm{St} \otimes \mathrm{St}'.$$

By Lemma 3.1 $\varepsilon_{C(s_2)} = 1$. Therefore we obtain from (4.2) the equations

$$R_{0,2} = \chi_{2,1} + \chi_{2,\mathrm{St}} + \chi_{2,\mathrm{St}'} + \chi_{2,\mathrm{St\,St}'},$$
$$R_{1,2} = \chi_{2,1} + \chi_{2,\mathrm{St}} - \chi_{2,\mathrm{St}'} - \chi_{2,\mathrm{St\,St}'},$$
$$R_{2,2} = \chi_{2,1} - \chi_{2,\mathrm{St}} + \chi_{2,\mathrm{St}'} - \chi_{2,\mathrm{St\,St}'},$$
$$R_{6,2} = \chi_{2,1} - \chi_{2,\mathrm{St}} - \chi_{2,\mathrm{St}'} + \chi_{2,\mathrm{St\,St}'}.$$

This system of linear equations has the unique solution given in assertion (a).

The unipotent irreducible characters of $\mathrm{SL}_3(q)$ are 1, St, and χ_{qs}. By Simpson and Frame [**13**, p. 492], and Proposition 2.2 they extend uniquely to unipotent irreducible characters of $C(s_4)$. Therefore the Green functions of $C(s_4)$ are given by

$$R^{C(s_4)}_{T_0,1} = 1 + 2\chi_{qs} + \mathrm{St},$$
$$R^{C(s_4)}_{T_1,1} = 1 - \mathrm{St},$$
$$R^{C(s_4)}_{T_3,1} = 1 - \chi_{qs} + \mathrm{St}.$$

By Lemma 3.1 $\varepsilon_{C(s_4)} = 1$. Therefore we obtain from (4.2) the equations

$$R_{0,4} = \chi_{4,1} + 2\chi_{4,qs} + \chi_{4,\mathrm{St}},$$
$$R_{1,4} = \chi_{4,1} - \chi_{4,\mathrm{St}},$$
$$R_{3,4} = \chi_{4,1} - \chi_{4,qs} + \chi_{4,\mathrm{St}}.$$

This system of linear equations has the unique solution given in (c). Using the same methods we obtain the assertions (h), (b), (d), (f) and (i).

By Corollary 7.3.5 of [**2**] $\varepsilon_{C(s_i)} R_{T,\hat{s}_i}$ is irreducible, if s_i is a regular element of G_σ. This completes the proof.

Proof of Table 4.4. The classification of the irreducible characters $\chi_{s,u}$ follows from (4.1) and Proposition 2.2. Their degrees are computed by means of (4.1) and Theorem 8.4.8 of [**2**].

The numbers of irreducible characters in each family $\chi_{s_i,u}$ equals the number $N(E_i, [w])$ of conjugacy classes of the semisimple elements s_i defined in Proposition 2.1. Let $w \in \Omega_{J_i}$ and

$$T(w, J_i) = \{ t \in T \mid w\sigma(t) = t,\ C_W(t) = W_{J_i} \}.$$

Define $Z_{J_i}(w) = \{ y \in \Omega_{J_i} \mid y^{-1} w\sigma(y) = w \}$. Then Lemma 3.1 of [**5**] asserts that $N(E_i, [w]) = |T(w, J_i)| / |Z_{J_i}(w)|$. As G is simply connected, Theorem 3.3 of [**5**] applies. Therefore $|T(w, J_i)| = F_{w,J_i}(q)$, where $F_{w,J_i}(X)$ is a polynomial with integral coefficients and degree $\deg(F_{w,J_i}(X)) = 4 - |J_i|$. In particular, $\deg(F_{1,\varnothing}(X)) = 4$.

TABLE 4.4. Degrees and numbers of irreducible characters of $^3D_4(q)$

character	degree	number, q even	number, q odd
1	1	1	1
$[\varepsilon_1]$	$q(q^4-q^2+1)$		
$[\varepsilon_2]$	$q^7(q^4-q^2+1)$		
St	q^{12}		
ρ_1	$\frac{1}{2}q^3(q^3+1)^2$		
ρ_2	$\frac{1}{2}q^3(q+1)^2(q^4-q^2+1)$		
$^3D_4[-1]$	$\frac{1}{2}q^3(q^3-1)^2$		
$^3D_4[1]$	$\frac{1}{2}q^3(q-1)^2(q^4-q^2+1)$		
$\chi_{2,1}$	q^8+q^4+1	0	1
$\chi_{2,\mathrm{St}}$	$q^3(q^8+q^4+1)$		
$\chi_{2,\mathrm{St}'}$	$q(q^8+q^4+1)$		
$\chi_{2,\mathrm{St}\,\mathrm{St}'}$	$q^4(q^8+q^4+1)$		
$\chi_{3,1}$	$(q+1)(q^8+q^4+1)$	$\frac{1}{2}(q-2)$	$\frac{1}{2}(q-3)$
$\chi_{3,\mathrm{St}}$	$q^3(q+1)(q^8+q^4+1)$		
$\chi_{4,1}$	$(q^3+1)(q^2-q+1)(q^4-q^2+1)$	$\frac{1}{2}(q^2+q)$	$\frac{1}{2}(q^2+q)$
$\chi_{4,\mathrm{St}}$	$q^3(q^3+1)(q^2-q+1)(q^4-q^2+1)$		
$\chi_{4,qs}$	$q(q^3+1)^2(q^4-q^2+1)$		
$\chi_{5,1}$	$(q^3+1)(q^8+q^4+1)$	$\frac{1}{2}(q^3-q^2-q-2)$	$\frac{1}{2}(q^3-q^2-q-3)$
$\chi_{5,\mathrm{St}}$	$q(q^3+1)(q^8+q^4+1)$		
χ_6	$(q+1)(q^3+1)(q^8+q^4+1)$	$\frac{1}{12}(q^4-4q^3+2q^2-2q+12)$	$\frac{1}{12}(q^4-4q^3+2q^2-2q+15)$
$\chi_{7,1}$	$(q-1)(q^8+q^4+1)$	$\frac{1}{2}q$	$\frac{1}{2}(q-1)$
$\chi_{7,\mathrm{St}}$	$q^3(q-1)(q^8+q^4+1)$		
χ_8	$(q-1)(q^3+1)(q^8+q^4+1)$	$\frac{1}{4}(q^4-2q)$	$\frac{1}{4}(q^4-2q+1)$
$\chi_{9,1}$	$(q^3-1)(q^2+q+1)(q^4-q^2+1)$	$\frac{1}{2}(q^2-q)$	$\frac{1}{2}(q^2-q)$
$\chi_{9,\mathrm{St}}$	$q^3(q^3-1)(q^2+q+1)(q^4-q^2+1)$		
$\chi_{9,qs'}$	$q(q^3-1)^2(q^4-q^2+1)$		
$\chi_{10,1}$	$(q^3-1)(q^8+q^4+1)$	$\frac{1}{2}(q^3-q^2+q)$	$\frac{1}{2}(q^3-q^2+q-1)$
$\chi_{10,\mathrm{St}}$	$q(q^3-1)(q^8+q^4+1)$		
χ_{11}	$(q+1)(q^3-1)(q^8+q^4+1)$	$\frac{1}{4}(q^4-2q^3)$	$\frac{1}{4}(q^4-2q^3+1)$
χ_{12}	$(q-1)^2(q^3+1)^2(q^4-q^2+1)$	$\frac{1}{24}(q^4+2q^3-q^2-2q)$	$\frac{1}{24}(q^4+2q^3-q^2-2q)$
χ_{13}	$(q+1)^2(q^3-1)^2(q^4-q^2+1)$	$\frac{1}{24}(q^4-2q^3-q^2+2q)$	$\frac{1}{24}(q^4-2q^3-q^2+2q)$
χ_{14}	$(q^6-1)^2$	$\frac{1}{4}(q^4-q^2)$	$\frac{1}{4}(q^4-q^2)$
χ_{15}	$(q-1)(q^3-1)(q^8+q^4+1)$	$\frac{1}{12}(q^4-2q^3+2q^2-4q)$	$\frac{1}{12}(q^4-2q^3+2q^2-4q+3)$

Using properties of the Brauer complex of G in [5] the first author introduced a method for finding the polynomial $F_{w,J_i}(X)$ for fixed $w \in \Omega_{J_i}$ and J_i. As an example we employ this method for the computation of $F_{1,\varnothing}(X)$, i.e., $w = 1 \in \Omega_{J_5} = W$.

The σ-conjugacy class [1] of 1 in $\Omega_\varnothing = W$ consists of 16 elements $w \in W$ by Proposition 2.1. Let $n = n(w)$ be the order of $w\tau^{-1}$. Since $0 \notin J_5 = \varnothing$ case (A) of [5] applies. Thus we have to find the number $k(1, \varnothing, q)$ of all $y \in Y \in \mathrm{Hom}(K^*, T)$ satisfying the following 16 systems of inequalities:

$$\sum_{i=0}^{n-1} q^i (w\tau^{-1})^{i+1}(r_j)(\tau^{-1}(y)) > 0 \quad \text{for } j = 1,2,3,4,$$

$$\sum_{i=0}^{n-1} q^i (w\tau^{-1})^{i+1}(r_0)(\tau^{-1}(y)) < q^n - 1.$$

By [5] $k(1, \varnothing, q) = F_{1,\varnothing}(q)$. Since $F_{1,\varnothing}(X)$ is an integral polynomial of degree 4, its coefficients are easily found by interpolation, if these numbers $k(1, \varnothing, q)$ can be determined for five different choices of q. In case q is odd, we get the following numbers:

q	3	5	7	9	11
$F_{1,\varnothing}(q)$	0	15	94	317	796

Thus interpolation yields that $F_{1,\varnothing}(X) = q^4 - 4q^3 + 2q^2 - 2q + 15$.

As $|Z_\varnothing(1)| = |Z_{J_5}(1)| = |W_\sigma| = 12$, it follows that

$$N(E_5, 1) = \tfrac{1}{12}(q^4 - 4q^3 + 2q^2 + 15),$$

which is the number of regular conjugacy classes of G_σ intersecting T_0 nontrivally.

For even q, we interpolate at $q = 2, 4, 8, 16$ and 32. Then the same method applies here as in all other remaining cases.

5. The blocks of irreducible characters. Let $r > 0$ be a prime number. In this section we determine the distribution of the irreducible characters $\chi_{s,u}$ of $G_\sigma = {}^3D_4(q)$ into r-blocks. As an application we then obtain the validity of R. Brauer's height zero conjecture, his conjecture on the number of irreducible characters in a block, and the Alperin-McKay conjecture for this class of simple groups.

Let R be a complete discrete rank one valuation ring with maximal ideal $\max(R) = \pi R$, residue class field $F = R/\pi R$ of characteristic $r > 0$, and quotient field $S = \operatorname{quot}(R)$ of characteristic 0 such that S and F are splitting fields for the finite group G. The block ideals of the r-block B of G in the group algebras FG, RG and SG are denoted by B, $\hat{B}$ and $B_S = \hat{B} \otimes_R S$ respectively. In particular, $B = \hat{B} \otimes_R F$. The number of simple SG-modules of B_S is denoted by $k(B)$, and $k_0(B)$ is the number of irreducible characters χ of G_σ belonging to B with height $\operatorname{ht}\chi = 0$. The number of irreducible modular characters of B is $l(B)$.

Let B be an r-block of a finite group G with defect group $\delta(B) =_G D$. Let $H = N_G(D)$ and $C = DC_G(D)$. By Brauer's first main theorem there is a unique block B_1 of H with defect group $\delta(B_1) = D$ such that $B = B_1^G$; it is called the Brauer correspondent of B in H.

The Alperin-McKay conjecture claims that $k_0(B) = k_0(B_1)$. Brauer conjectured that in general $k(B) \leqslant |D|$, and his height zero conjecture says that $k_0(B) = k(B)$ if and only if $\delta(B)$ is abelian.

Let B be an r-block of a finite group G with defect group $\delta(B) =_G D$. Then by Brauer's extended first main theorem there is a block b of $C = DC_G(D)$ with defect group D such that $B = b^G$. Any such block b of C is called a root of B. By Corollary 4.6 of [7, p. 204], b contains exactly one irreducible character χ_s which has D in its kernel. This character χ_s of b is called the canonical character of the block B. If $H = N_G(D)$, then χ_s is uniquely determined by B up to H-conjugacy. The inertial subgroup $T_H(b) = \{x \in H \mid b^x = b\} = T_H(\chi_s) = \{x \in H \mid \chi_s^x = \chi_s\}$.

By Theorem 4.3 every irreducible character χ of $G_\sigma = {}^3D_4(q)$ is of the form $\chi_{t,u}$, where t is a representative of a semisimple conjugacy class of G_σ, and where χ_u is an irreducible unipotent character of $C_{G_\sigma}(t)$. We now study the distribution of the irreducible characters of G_σ into r-blocks B of G_σ. Such a block B is called *unipotent*, if B contains a unipotent character of G_σ.

COROLLARY 5.1. (a) *For every prime number $r > 0$, $G_\sigma = {}^3D_4(q)$ contains r-blocks of defect zero.*

(b) *If $r \neq 2$, then G_σ contains unipotent r-blocks of defect zero.*

PROOF. Let r^a be the order of a Sylow r-subgroup of G_σ. Then by Lemma 4.19 of [7, p. 159], an irreducible character χ of G_σ belongs to an r-block B with defect $d(B) = 0$ if and only if $r^a \mid \chi(1)$. Hence (b) follows immediately from Table 4.4.

If $r \mid q$, then (a) holds by Steinberg's tensor product theorem. Let $r \nmid q$. From (b) it follows that we may assume that $r = 2$. Then by Table 4.4 the $\frac{1}{4}(q^4 - q^2)$ irreducible characters χ_{14} yield 2-blocks of defect zero.

LEMMA 5.2. *Let r be a prime number, $r \notin \{2, 3, p\}$. If $D \neq 1$ is a defect group of an r-block B of G_σ, then either D is cyclic or a Sylow r-subgroup which is abelian and generated by 2 elements.*

PROOF. As $r \notin \{2, 3, p\}$, G_σ has an abelian Sylow r-subgroup S by Corollary 5.19 of [1, p. 212]. Proposition 1.2 asserts that S has at most two generators.

If D is not cyclic, then by Theorem 9.2 of [7, p. 231], there is a central element $1 \neq x \in D$ and an r-block b of $C_{G_\sigma}(x)$ such that $B = b^{G_\sigma}$ and both blocks B and b have defect group D.

Using the character tables of $SL_2(q)$, $SL_3(q)$, and $SU_3(q)$ given in [6 and 13] it is easy to see that each r-block b_1 of any of these groups has a Sylow r-subgroup as defect group $\delta(b_1)$, if $\delta(b_1)$ is not cyclic. Therefore Proposition 2.2 implies that $\delta(b) =_{G_\sigma} D$ is a Sylow r-subgroup of G_σ. This completes the proof.

PROPOSITION 5.3. *Let q be odd, and let 2^a be the highest power of 2 dividing $q - 1$ or $q + 1$ if $q \equiv 1(4)$ or $q \equiv 3(4)$, respectively. Let Q be a Sylow 2-subgroup of $SL_2(q)$ contained in a Sylow 2-subgroup P of G_σ, and let Z be a cyclic 2-subgroup of Q of order $|Z| = 2^a$. Then P contains an involution x such that $S = \{Q, x\}$ is a semidihedral subgroup of P of order $|S| = 2^{a+2}$.*

If B is a 2-block of G_σ with defect group $D \neq 1$, then one of the following holds:

(a) *$D =_{G_\sigma} P$ if and only if $B = B_0$ the principal 2-block of G_σ.*

(b) *$D \simeq S * Z$.*

(c) *$D \simeq S$.*

(d) *$D \simeq Z \times Z$.*

(e) *D is isomorphic to a Klein four subgroup of P.*

(f) *D is a cyclic Sylow 2-subgroup of a cyclic maximal torus of G_σ.*

PROOF. G_σ has only one class of involutions $s_2 \neq 1$ by Proposition 2.1. Let $C = C_{G_\sigma}(s_2)$. Then by Proposition 2.2 C' is the central product $SL_2(q^3) * SL_2(q)$. Hence a Sylow 2-subgroup P of G_σ contains a central product of two isomorphic generalized quaternion groups Q, and $|P : Q * Q| = 2$.

By Proposition 2.2 the defect group D of B may be chosen such that $s_2 \in Z(D)$. Thus $K = DC_{G_\sigma}(D) \leqslant_G C$. Let b be a root of B in K. Then $\delta(b) =_K D$ and $B = b^{G_\sigma}$. Furthermore, $B_1 = b^C$ exists by Lemma 6.1 of [7, p. 209], and $\delta(B_1) = D$, because $B = B_1^{G_\sigma} = (b^C)^{G_\sigma} = b^{G_\sigma}$.

By the character table of $U = \mathrm{SL}_2(q)$ [**6**, p. 228], we know that the principal 2-block $B_0(U)$ of U is the only 2-block of U with defect group Q, and that all other 2-blocks of U have either the center $Z(Q)$ of order $|Z(Q)| = 2$ or a cyclic group Z of order $2^a \geqslant 4$ as a defect group. Up to isomorphism Q is also a Sylow 2-subgroup of $\mathrm{SL}_2(q^3)$. As $C' = \mathrm{SL}_2(q) * \mathrm{SL}_2(q^3)$, each block A of C' with defect group $\delta(A) = E$ is mapped onto a block $\tau(A)$ of $\bar{C} = C'/\{s_2\} \simeq \mathrm{PSL}_2(q) \times \mathrm{PSL}_2(q^3)$ with defect group $\delta(\tau(A)) = E/\{s_2\}$. Hence E is isomorphic to one of the 2-subgroups $Q * Q$, $Q * Z$, Q, $Z * Z$, Z, or $Z(Q)$. Since $|C : C'| = 2$ Green's theorem [7, p. 107] implies that every 2-block A of C' induces up to a 2-block $B' = A^C$ of C. By Theorem 3.14 of [7, p. 201] $\delta(B') \cap C' = \delta(A) = E$. Using now the group structure of P and C the assertions (b)–(f) follow.

Furthermore, by Brauer's third main theorem the principal 2-block B_0 of G is the only 2-block of highest defect.

PROPOSITION 5.4. *Let $3 \nmid q$. If B is a 3-block of G_σ with defect group D, then one of the following statements holds*:

(a) *D is nonabelian if and only if D is a Sylow 3-subgroup of G_σ.*

(b) *D is a noncyclic Sylow 3-subgroup of a maximal torus of G_σ.*

(c) *D is a cyclic Sylow 3-subgroup of a cyclic maximal torus of G_σ.*

PROOF. (a) Let B be a 3-block of G_σ with a nonabelian defect group D. Either $3 \mid q - 1$ or $3 \mid q + 1$. Suppose that $3 \mid q - 1$. By Theorem 9.2 of [7, p. 231], there is a central element $1 \neq x \in D$ of order 3 and an r-block b of $C = C_{G_\sigma}(x)$ such that $B = b^{G_\sigma}$, and D is a defect group of b. By Proposition 2.2, $C' = \mathrm{SL}_3(q) * Z$, where Z is a cyclic group with $|Z| = q^2 + q + 1$, and where the cyclic group C/C' of order 3 acts trivially on Z. By the character table of [**13**, p. 487], only the principal 3-block of $U = \mathrm{SL}_3(q)$ has a Sylow 3-subgroup D_1 of U as a defect group. Since 3 divides $|Z|$ only to the first power, C' has $\frac{1}{3}(q^2 + q + 1)$ blocks b_1 with defect group D_1, and all other blocks of C' have abelian or cyclic defect groups. As C/C' acts trivially on Z, it follows from Theorem 3.14 of [7, p. 201], that each 3-block $(b_1)^C$ of C has a Sylow 3-subgroup P of C and thus of G_σ as a defect group. By Proposition 2.2 all other blocks b_2 of C have abelian or cyclic defect groups. Hence $D =_{G_\sigma} P$ and $b \in \{(b_1)^C\}$ by Lemma 9.1 of [7, p. 230].

Because of Proposition 2.2 the same argument can also be applied in the case of $3 \mid q + 1$. The converse implication is trivial.

So we may assume that D is abelian. The assertions (b) and (c) follow from Propositions 1.2 and 2.2 and the definition of a defect group, see [7, pp. 126 and 231].

LEMMA 5.5. *Let T_σ be a maximal torus and D a Sylow r-subgroup of G_σ contained in T, where $r \neq p$. Let $s \in T_\sigma$ be an r'-element of T_σ, and let $\hat{s}$ denote the linear character* *of T corresponding to s. Then for every $y \in D$ the Deligne-Lusztig characters $R_{T,\widehat{sy}}$ and $R_{T,\hat{s}}$ agree on all r'-elements $x \in G_\sigma$.*

PROOF. Let the r'-element $x = tu \in G_\sigma$ be in Jordan form, where t is semisimple and u is unipotent. By Lemma 3.2

$$R^G_{T,\widehat{sy}}(x) = \begin{cases} \dfrac{\varepsilon_{C(t)}\varepsilon_T |C(t)|_{p'}}{|T|}\widetilde{sy}(t), & \text{if } u = 1, \text{ and } t \in_{G_\sigma} T, \\ \widetilde{sy}(t) Q_T^{C(t)}(u), & \text{if } u \neq 1, \text{ and } t \in_{G_\sigma} T, \\ 0, & \text{if } t \notin_{G_\sigma} T, \end{cases}$$

where

$$\widetilde{sy}(t) = \frac{1}{|C_{W(T)}(t)|} \sum_{w \in W(T)} \widehat{sy}(wtw^{-1}),$$

and $W(T) = N_{G_\sigma}(T)/T$. As $t' = wtw^{-1} \in T$ is an r'-element for each $w \in W(T)$, $\hat{y}(t') = 1$ for each $y \in D$, because D is a Sylow r-subgroup of T. Therefore

$$\begin{aligned}\widetilde{sy}(t) &= \frac{1}{|C_{W(T)}(t)|} \sum_{w \in W(T)} \hat{s}(wtw^{-1})\hat{y}(wtw^{-1}) \\ &= \frac{1}{|C_{W(T)}(t)|} \sum_{w \in W(T)} \hat{s}(wtw^{-1}) = \tilde{s}(t).\end{aligned}$$

Hence $R^G_{T,\widehat{sy}}(x) = R^G_{T,\hat{s}}(x)$.

PROPOSITION 5.6. *Let B be an r-block of G_σ with a cyclic defect group $\delta(B) =_{G_\sigma} D \neq 1$. Then the following assertions hold*:

(a) *D is a Sylow r-subgroup of a cyclic maximal torus T of G_σ such that $D \leqslant T \leqslant C = C_{G_\sigma}(D)$.*

(b) *B is either the principal r-block of G_σ or B determines (up to G_σ-conjugacy) uniquely a regular r'-element s of G_σ contained in T such that $\theta = \varepsilon_C\varepsilon_T R^C_{T,\hat{s}}$ is the canonical character of a root b of B in C.*

(c) *B is the principal r-block of G_σ if and only if D is the Sylow r-subgroup of the Coxeter torus T_5 of G_σ, and B has $\frac{1}{4}(|D| - 1)$ exceptional irreducible characters χ_y with $1 \neq y \in D$, and 4 nonexceptional irreducible characters which are the 4 unipotent characters 1, St, ρ_1 and ${}^3D_4[-1]$.*

(d) *If B is a nonprincipal r-block of G_σ, then an irreducible character $\chi_{t,u}$ of G_σ belongs to B if and only if $t \sim_{G_\sigma} sy$ for some $y \in D$, and χ_u is a unipotent irreducible character of $C_{G_\sigma}(sy)$ such that $\widehat{sy}\chi_u$ belongs to an r-block β of $C_{G_\sigma}(sy)$ with $B = \beta^G$.*

If $y \neq 1$, then sy is regular in G_σ and $\chi_{t,u} = \chi_{sy}$ is an exceptional character of B. If s is regular in G_σ, then χ_s is the only nonexceptional character of B. Otherwise $\chi_{s,1}$ and $\chi_{s,St}$ are the nonexceptional characters of B.

PROOF. By Humphreys' theorem **[10]** $r \nmid q$. So, if $q = 2$, then $r \in \{3, 7, 13\}$. In each case $|D| = r$, and all assertions are easily verified by means of Propositions 1.2, 2.2 and Table 4.4. Thus we may assume also that $q \neq 2$. Let e be the smallest integer such that $r \mid q^e - 1$. Then $e \in \{1, 2, 3, 6, 12\}$.

(a) By Propositions 5.3 and 5.4 we may assume that $r \notin \{2,3\}$. Let $C = C_{G_\sigma}(D)$. By Dade's theorem [7, p. 270], there is an r-block b of C with defect group D such that $B = b^G$. Then by Lemma 5.2 and Corollary 5.19 of [1, p. 212], D is contained in a maximal torus T of G_σ such that $T \leqslant C = C_{G_\sigma}(D)$. Let $x \neq 1$ be a generator of D.

If x is regular in G_σ, then $C_{G_\sigma}(x) = T$, and b is an r-block of T with defect group D. Hence D is a Sylow r-subgroup of T. Thus T is a cyclic Coxeter torus T_5 of G_σ by Proposition 1.2.

Suppose that x is not regular. Let $e = 1$. Then by Proposition 2.2 there is an r-block b_1 of $C_1 = C_{G_\sigma}(s_3)$ or of $C_2 = C_{G_\sigma}(s_5)$ with defect group $\delta(b_1) = D$, because $r \neq 2$, and $x \in S_\sigma \cong \mathbf{Z}_{q-1}$ or $x \in S_\sigma \cong \mathbf{Z}_{q^3-1}$. Since all r-blocks of $\mathrm{SL}_2(q^k)$, $k \in \{3,1\}$, either have a Sylow r-subgroup as a defect group or have defect zero, it follows that D is a Sylow r-subgroup of $S_\sigma = \mathbf{Z}_{q-1}$ or of $S_\sigma \cong \mathbf{Z}_{q^3-1}$. Therefore D is contained in the maximal torus $T_2 \cong \mathbf{Z}_{(q^3+1)(q-1)}$ or $T_1 \cong \mathbf{Z}_{(q^3-1)(q+1)}$ of $C = C_{G_\sigma}(D)$. Hence D is a Sylow r-subgroup of the cyclic maximal torus T_2 or T_1.

Similarly one can show that D is the Sylow r-subgroup of T_1 or T_2 for $e \in \{2,3,6\}$. Hence (a) holds.

Let $e = 1$ and $T = T_2$. Suppose that $r \neq 2$. Then $C \cong C_{G_\sigma}(s_3)$, where D is a Sylow r-subgroup of the central torus S_σ of C with order $|S_\sigma| = q - 1$. The canonical character Θ of the root b of B in C has D in its kernel $\ker\Theta$ by [7, p. 205]. Furthermore, Θ is projective as an irreducible character of C/D by Brauer's extended first main theorem. Since the centralizer of all noncentral semisimple elements of $\mathrm{SL}_2(q^3)$ are cyclic, it follows from the character table [6, pp. 228 and 235], and Proposition 2.2 that there is a regular r'-element s of C in T with order $o(s) \mid q^3 + 1$ such that $\Theta = \varepsilon_{C(D)}\varepsilon_T R^C_{T,s}$, because $e = 1$. As $D \leqslant \ker\Theta$, Theorem 7.2.8 of [2] implies that $D \leqslant \ker\hat{s}$. Furthermore, $s \in \{s_{10}, s_9, s_{11}\}$ up to G_σ-conjugacy.

If $e = 1$, and $T = T_1$, then the same argument shows that $\Theta = \varepsilon_{C(D)}\varepsilon_T R^C_{T,\hat{s}}$, where $s = s_8$ is a regular r'-element of G_σ with order $o(s) \mid q + 1$.

Suppose that $e = 2$, and $T = T_1$. Then there exists a regular r'-element s of C such that $\Theta = \varepsilon_{C(D)}\varepsilon_T R^C_{T,\hat{s}}$ is the canonical character of b, where $o(s) \mid q^3 - 1$. Hence $s \in \{s_5, s_8, s_4\}$. If $e = 2$, and $T = T_2$, then by the same argument $s = s_{11}$.

Let $e = 3$. Then $r \mid q^2 + q + 1$, and $C = C_{G_\sigma}(s_4)$ by Proposition 2.2. Furthermore, $T = T_1$. By the character table [13] of $\mathrm{SL}_3(q)$ there is a regular r'-element s of C contained in T such that $\Theta = \varepsilon_{C(D)}\varepsilon_T R^C_{T,\hat{s}}$ is the canonical character of b, where $o(s) \mid q^2 - 1$. Hence $s \in \{s_8, s_7\}$.

If $e = 6$, then the same argument shows that $s \in \{s_3, s_{11}\}$, $T = T_2$.

Let $e = 12$. Then D is a Sylow r-subgroup of the cyclic Coxeter torus $T = T_5$. Its elements $t \neq 1$ are regular by Table 4.4. In particular, $C = C_{G_\sigma}(D) = T$. If B is not the principal r-block, then the canonical character Θ of the root b of B in C is of the form $\Theta = \hat{s}$, where s is a regular r'-element of T. So B is the principal r-block if and only if D is a Sylow r-subgroup of T_5 and $s = 1$.

If $r = 2$, then Proposition 5.3 asserts that the generator x of D has order 2^{a+1}, and that D is a Sylow 2-subgroup of a cyclic maximal torus T. Hence x is regular by

Propositions 2.1 and 2.2, and $C = C_{G_\sigma}(D) = T$. Therefore the canonical character Θ of the root b of B is of the form $\Theta = \hat{s}$, where $s \in T$ is a regular r'-element of G, because B has inertial index one by Dade's theorem. Thus (b) holds.

Suppose that B is a nonprincipal r-block. Then in any case for $e \in \{1,2,3,5,12\}$ we have shown that the canonical character Θ of a root b of B in $C = C_{G_\sigma}(D)$ is of the form $\Theta = \varepsilon_{C(D)}\varepsilon_T R^C_{T,\hat{s}}$, where s is a regular r'-element of C contained in a cyclic torus T. Furthermore, $D \leqslant \ker \hat{s}$, where $\hat{s}$ is the linear character of T corresponding to $s \in T$. By Propositions 2.1 and 2.2 sy is a regular element of G_σ for each $1 \neq y \in D$. Hence each $\chi_{sy} = \varepsilon_T R_{T,\widehat{sy}}$ with $1 \neq y \in D \leqslant T$ is an irreducible character of G_σ by Lemma 3.1 and Corollary 7.3.5 of [**2**].

Two irreducible characters of G_σ belong to the same r-block of G_σ if they agree on all r'-elements, see [7, p. 150 and p. 179]. Thus all irreducible characters χ_{sy}, $1 \neq y \in D$, belong to one r-block B_1 of G_σ by Lemma 5.5. Let D_1 be its cyclic defect group. For $x \in D$ $\chi_{sy}(x) = \varepsilon R_{T,\hat{s}}(x) = \varepsilon\varepsilon_{C(x)}\varepsilon_T|C(x)|_{p'}\tilde{s}(x)$ by Lemmas 3.2 and 5.5, where

$$\tilde{s}(x) = \frac{1}{|C_{W(T)}(x)|}\sum_{w \in W(T)} \hat{s}(wxw^{-1}).$$

As $D \leqslant \ker \hat{s}$, $\tilde{s}(x) = |W(T): C_{W(T)}(x)| \neq 0$ by Lemmas 3.4 and 3.5. Thus $\chi_{sy}(x) \neq 0$ for every $x \in D$, and $D =_{G_\sigma} D_1$ by Lemma 59.5 of [**6**] and (a). Let b_1 be a root of B_1 in $C = C_{G_\sigma}(D)$. As shown above there is a regular element s_1 of T such that the canonical character of b_1 is of the form $\Theta_1 = \varepsilon_C\varepsilon_T R^C_{T,\hat{s}_1}$. Let $\mathscr{H}$ be the Brauer homomorphism from the center ZFG_σ into ZFC with respect to D. Let ω_s be the central character of χ_{sy}, and ω_{s_1} the one of Θ_1. As $B_1 = b_1^{G_\sigma}$ it follows from Brauer's extended first main theorem that $\omega_s = \omega_{s_1}\mathscr{H}$ on ZFG_σ.

From [**7**, p. 144], we obtain for every r'-element $x \in T$ that

$$\frac{|x^{G_\sigma}|\chi_{sy}(x)}{\chi_{sy}(1)} \equiv \frac{|x^C|\Theta_1(x)}{\Theta_1(1)} \mod \pi R.$$

Applying Lemma 3.2 and Theorem 7.5.1 of [**2**] we get

$$\frac{|x^{G_\sigma}|\varepsilon_{C_{G_\sigma}(x)}\varepsilon_T|C_G(x)|_{p'}s(x)}{|T|\,|G_\sigma:T|_{p'}} \equiv \frac{|x^C|\varepsilon_{C_C(x)}\varepsilon_T|C_C(x)|_{p'}s_1(x)}{|T|\,|C:T|_{p'}}.$$

Hence $\varepsilon_{C_{G_\sigma}(x)}|G_\sigma: C_{G_\sigma}(x)|_p s(x) \equiv \varepsilon_{C_C(x)}|C: C_C(x)|_p s_1(x)$. Corollary 6.5.7 of [**2**] and Proposition 2.2 assert that $\varepsilon_{C_{G_\sigma}(x)} = \varepsilon_{C_C}(x)$ for all r'-elements $x \in T$. Let $C_1 = C_{W(T)}(x)$, and $W_1 = W(T)$. Then

$$\tilde{s}(x) = \frac{1}{|C_1|}\sum_{w \in W_1} \hat{s}(wxw^{-1}) \equiv s_1(x) = \frac{1}{|C|}\sum_{w \in W} \hat{s}_1(wxw^{-1}).$$

Since $(|T/D|, r) = 1$, and D is in the kernel of s and s_1, it follows that $\hat{s}$ and $\hat{s}_1$ are $W(T)$-conjugate. Hence θ and θ_1 are $N_{G_\sigma}(D)$-conjugate, because $W(T) = N_{G_\sigma}(D)/C_{G_\sigma}(D)$ by Proposition 1.2. Therefore $B = B_1$. Let $t = |C_W(s)|$. Then t is the inertial index of B, and it follows from Dade's theorem [7, p. 177] that the $(|D| - 1)t^{-1}$ irreducible characters χ_{sy}, $1 \neq y \in D$, are the exceptional characters of B.

If s is regular in G_σ, then $\chi_s = \varepsilon_T R_{T,\hat{s}}$ is an irreducible character of G_σ, which by the previous argument belongs to B. By Lemma 3.4 $t = 1$. Hence by Dade's theorem χ_s is the only nonexceptional character of B.

Suppose that s is not regular. Let $e = 1$. Then $T = T_2$ and $s \in \{s_{10}, s_9\}$ by the proof of (b). B has inertial index $t = 2$ by Lemma 3.4. Let $s = s_{10}$. Then by Theorem 4.3

$$\chi_{s,1} = -\tfrac{1}{2}(R_{2,10} + R_{6,10}), \quad \chi_{s,\mathrm{St}} = -\tfrac{1}{2}(R_{2,10} - R_{6,10}).$$

Using Proposition 2.2 and Tables 3.6 and 4.4 it follows that $|x^{G_\sigma}|R_{6,10}(x)/\chi_{s,u}(1) \equiv 0$ mod πR for every r'-element x of G_σ. Let $1 \neq y \in D$. Then χ_{sy} belongs to B. Since $e = 1$, $v = \chi_{sy}(1)/2\chi_{s,u}(1) \equiv 1$ mod πR by Table 4.4. $\chi_{sy}(x) = -R_{2,s}(x)$ for every r'-element $x \in G_\sigma$ by Theorem 4.3 and Lemma 5.5. Hence

$$\frac{|x^{G_\sigma}|\chi_{s,u}(x)}{\chi_{s,u}(1)} \equiv \frac{|x^{G_\sigma}|[-R_{2,s}(x)]}{2\chi_{s,u}(1)} \equiv v\frac{|x^{G_\sigma}|\chi_{sy}(x)}{\chi_{sy}(1)} \equiv \frac{|x^{G_\sigma}|\chi_{sy}(x)}{\chi_{sy}(1)}.$$

Therefore $\chi_{10,1}$ and $\chi_{10,\mathrm{St}}$ are the two nonexceptional characters of B. Now let $s = s_9$. Then by Theorem 4.3

$$\chi_{s,1} = -\tfrac{1}{6}(R_{6,9} + 3R_{2,9} + 2R_{4,9}), \qquad \chi_{s,\mathrm{St}} = \tfrac{1}{6}(R_{6,9} - 3R_{2,9} + 2R_{4,9}).$$

Using Proposition 2.2 and Tables 3.6 and 4.4 it follows that

$$\frac{|x^{G_\sigma}|R_{6,9}(x)}{\chi_{s,u}(1)} \equiv 0 \equiv \frac{|x^{G_\sigma}|R_{4,9}(x)}{\chi_{s,u}(1)} \mod \pi R$$

for every r'-element x of G_σ. Applying the above argument again we see that $\chi_{9,1}$ and $\chi_{9,\mathrm{St}}$ are the nonexceptional characters of B.

Let $e = 3$. Then $T = T_1$ and $s = s_7$ by the proof of b). Now

$$\chi_{s,1} = -\tfrac{1}{2}(R_{1,7} + R_{6,7}), \qquad \chi_{s,\mathrm{St}} = -\tfrac{1}{2}(R_{1,7} - R_{6,7})$$

by Theorem 4.3. Using Proposition 2.2 and Tables 3.6 and 4.4 it follows that for every r'-element $x \in G_\sigma$

$$\frac{|x^{G_\sigma}|R_{6,7}(x)}{\chi_{s,u}(1)} \equiv 0 \mod \pi R.$$

Let $1 \neq y \in D$. Then χ_{sy} belongs to B. Since $e = 3$,

$$v = \frac{\chi_{sy}(1)}{2\chi_{s,u}(1)} = \frac{q^3+1}{2q^n} \equiv 1 \mod \pi R$$

by Table 4.4, where $n \in \{0, 3\}$. From Theorem 4.3 and Lemma 5.5 follows that $\chi_{sy}(x) = -R_{1,s}(x)$ for every r'-element x of G_σ. Hence

$$\frac{|x^{G_\sigma}|\chi_{s,u}(x)}{\chi_{s,u}(1)} \equiv \frac{|x^{G_\sigma}|[-R_{1,s}(x)]}{2\chi_{s,u}(1)} \equiv v\frac{|x^{G_\sigma}|\chi_{sy}(x)}{\chi_{sy}(1)} \equiv \frac{|x^{G_\sigma}|\chi_{sy}(x)}{\chi_{sy}(1)}.$$

Therefore χ_7 and $\chi_{7,\mathrm{St}}$ are the two nonexceptional characters of B.

Replacing q by $-q$ the cases $e = 2, 6$ follow from the cases $e = 1,3$, respectively.

Let β be the r-block of $C_{G_\sigma}(s)$ containing the unipotent characters $\hat{s}\chi_u$ corresponding to $\chi_{s,u}$ of B. Then $B = \beta^{G_\sigma}$ by [7, p. 136], because the linear character $\hat{s}$ of $T = C_{G_\sigma}(s) \cap C_{G_\sigma}(D)$ is the canonical character of β. Hence (d) holds.

Finally let B be the principal r-block. Then D is the Sylow r-subgroup of the Coxeter torus T_5. Therefore every element $1 \neq y \in D$ is regular by Propositions 2.1 and 2.2, and B has inertial index $t = 4$ by Proposition 1.2. Hence B has $\frac{1}{4}(|D| - 1)$ irreducible nonexceptional characters χ_y with $1 \neq y \in D$ by Lemma 5.5 and the proof of (b). Furthermore, the following 4 unipotent irreducible characters 1, St, ρ_1 and ${}^3D_4[-1]$ belong to B by Table 4.4. Hence by Dade's theorem we have found all the characters of B. This completes the proof.

LEMMA 5.7. *Let $r \notin \{p, 2, 3\}$, and let B be an r-block of $G_\sigma = {}^3D_4(q)$ with a noncyclic defect group D. Let $H = N_{G_\sigma}(D)$. Then:*

(a) *$C_{G_\sigma}(D) = T$ is a maximal torus of G_σ.*

(b) *Up to G_σ-conjugacy there exists a unique r'-element $s \in T$ and a root b of B in $C_{G_\sigma}(D) = T$ such that the linear character $\hat{s}$ of T is the canonical character of B.*

(c) *Let $W(T)$ be the Weyl group of T. Then the inertial subgroup $T_H(b) = T(D \cdot C_{W(T)}(s))$, where $D \cdot C_{W(T)}(s)$ denotes the split extension of D by $C_{W(T)}(s)$ induced by the action of $W(T)$ on T.*

(d) *If B_1 is the Brauer correspondent of B in H, then its r-adic block ideal $\hat{B}_1$ is Morita equivalent to the group algebra $R[D \cdot C_{W(T)}(s)]$, and $k(B_1) = k_0(B_1) = k(S[DC_{W(T)}(s)]) \leqslant |D|$.*

PROOF. (a) As D is not cyclic, Lemma 5.2 asserts that D is an abelian Sylow r-subgroup of G_σ. By Corollary 5.19 of [1, p. 212] and Proposition 2.2, $C_{G_\sigma}(D) = T$ is a maximal torus of G_σ.

(b) Let the r-block b of $C_{G_\sigma}(D) = T$ be a root of B, and let $\Theta \in \mathrm{Irr}_S(b)$ be the canonical character of B. Certainly Θ is a linear character of T. As $D \subseteq \ker_\Theta$ there is up to H-conjugacy a unique element s of T such that $\Theta = \hat{s} \in \mathrm{Irr}_S(T)$. As $r \nmid |T:D|$, s is clearly an r'-element. Furthermore, $\mathrm{Irr}_S(b) = \{\widehat{sy} \mid y \in D\}$.

(c) The inertial subgroups of b and χ are given by

$$T_H(b) = T_H(\Theta) = T\big(D \cdot C_{W(T)}(s)\big),$$

because $W(T) = N_{G_\sigma}(T)/T = N_{G_\sigma}(D)/T$. Since $r \notin \{2, 3, p\}$, it follows from Proposition 1.2 and the lemma of Schur and Zassenhaus that $D \cdot C_{W(T)}(s)$ is the split extension of D by $C_{W(T)}(s)$ induced by the action of $W(T)$ on T.

(d) Let $B' = b^{T_H(b)}$ be the block of $T_H(b)$ with the same block idempotent as b, and let $\hat{B}$ be its r-adic block ideal. If $\hat{B}_1$ denotes the r-adic block ideal of the Brauer correspondent B_1 of B in H, then by Theorem 2.5 of [7, p. 197], the algebras $\hat{B}$ and $\hat{B}_1$ are Morita equivalent, and $k(B') = k(B_1)$, $k_0(B') = k_0(B_1)$.

It is easy to see that $\hat{B}' \simeq R[D \cdot C_{W(T)}(s)]$. By Lemmas 3.4 and 3.5 $|C_{W(T)}(s)|$ divides 24. As D is abelian and $r \notin \{2, 3\}$, it follows that

$$k(B') = k_0(B') = k\big(S\big[D \cdot C_{W(T)}(s)\big]\big).$$

Hence $k(B_1) = k_0(B_1) \leqslant |D|$, the latter inequality follows by application of Lemmas 3.4 and 3.5 and the structure of the group algebra $S[D \cdot C_{W(T)}(s)]$.

PROPOSITION 5.8. *Let B be an r-block of G_σ with a noncyclic abelian defect group $\delta(B) =_{G_\sigma} D$. Then the following assertions hold*:

(a) $C_{G_\sigma}(D) = T$ *is a maximal torus of* G_σ, *and D is a Sylow r-subgroup of T.*

(b) *Up to G_σ-conjugacy there exists a unique r'-element $s \in T$ and a root b of B in $C_{G_\sigma}(D) = T$ such that the linear character $\hat{s}$ of T is the canonical character of B.*

(c) *If $H = N_{G_\sigma}(D)$, and $T_H(b)$ denotes the inertial subgroup of b in H, then $T_H(b)/T \simeq C_{W(T)}(s)$, where $W(T) = N_{G_\sigma}(T)/T$.*

(d) *An irreducible character $\chi_{t,u}$ of G_σ belongs to B if and only if $t \sim_{G_\sigma} sy$ for some $y \in D$, and χ_u is a unipotent irreducible character of $C_{G_\sigma}(sy)$ such that $\widehat{sy}\chi_u$ belongs to an r-block β of $C_{G_\sigma}(sy)$ with $B = \beta^{G_\sigma}$.*

(e) *B is the principal r-block of G_σ if and only if $s = 1$ and $r \geqslant 5$.*

(f) *The number $l(B)$ of modular irreducible characters of B equals the number of unipotent irreducible characters of $C_{G_\sigma}(s)$, provided $s \neq 1$.*

PROOF. By Humphreys' theorem [**10**] $r \nmid q$. If $r \neq 3$, then assertions (a), (b), and (c) hold by Proposition 5.3 and Lemmas 5.7 and 5.2.

Let $r = 3$. Then by Lemma 5.4 D is a Sylow 3-subgroup of a maximal torus T of G_σ. As D is not cyclic, T is isomorphic to T_0, T_3, T_4, or T_6 by Proposition 1.2. From Corollary 5.19 of [**1**, p. 212], and Proposition 2.2 it follows that $C_{G_\sigma}(D) = T$. Thus (a) holds, and (b) can now be shown as in Lemma 5.7. If $\hat{s} \in \hat{T}$ denotes the canonical character of B, then $T_H(b) = T_H(\hat{s}) = \{h \in H \mid s^h = s\}$. Hence (c) holds also for $p = 3$.

(e) is a consequence of (b) and Brauer's third main theorem, because by Propositions 5.3 and 5.4 we may assume that $r \geqslant 5$.

(d) Fix $s \in T = C_{G_\sigma}(D)$ such that its corresponding linear character $\hat{s}$ of T is the canonical character of a block b of $C_{G_\sigma}(D)$ satisfying $B = b^{G_\sigma}$. Then s is uniquely determined by B up to G_σ-conjugation and $\mathrm{Irr}_S(b) = \{\widehat{sy} \mid y \in D\}$. Furthermore, $D \leqslant \ker \hat{s}$.

Let $\mathscr{H}$ be the Brauer homomorphism with respect to D from ZFG_σ into $ZFC_{G_\sigma}(D)$. As $T = C_{G_\sigma}(D)$ and $B = b^{G_\sigma}$, Brauer's extended first main theorem implies that on each r-regular conjugacy class x^{G_σ} of G_σ with defect group D the central linear character λ of B agrees with $\tau(x) = 1/|C_{\overline{H}}(x)|\sum_{w \in \overline{H}} \hat{s}(wxw^{-1})$, where $\overline{H} = H/T$. Since by (a) D is a Sylow r-subgroup of T it follows from Proposition 1.2 that $W(T) = H/T$. Using now the notation introduced before Table 3.6 we obtain $\tau(x) = \mathscr{N}_{T,\hat{s}}(x)$. Therefore by [7, p. 144], an irreducible character $\chi_{t,u}$ of G_σ belongs to B if and only if

$$(*) \qquad \frac{|x^{G_\sigma}|\chi_{t,u}(x)}{\chi_{t,u}(1)} \equiv \mathscr{N}_{T,\hat{s}}(x) \bmod \pi R \text{ for every } r'\text{-element } x \text{ of } T$$

with defect group D.

Suppose that the irreducible character $\chi_{t,u}$ of G_σ belongs to B. If $f = f^2 \neq 0$ denotes the block idempotent of B, then $\lambda(f) = 1 \in F$. By Lemma 7.2 of [7, p. 179] f is a linear combination of r-regular class sums of G_σ. Therefore $\chi_{t,u} \neq 0$ for some

r'-element x. Hence $t \in T$ (up to G_σ-conjugacy) by Lemma 3.2 and Theorem 4.3. Let $t = zy = yz$, where $z \in T$ is r-regular and $y \in D$. By Lemma 5.5, $R_{T,\widehat{zy}}$ and $R_{T,\hat{z}}$ agree on all r'-elements $x \in G_\sigma$, and D is in the kernel of $\hat{z} \in \hat{T}$.

As $q \neq 2$, we obtain from Theorem 4.3, Table 3.6, and Lemma 3.4 that for all unipotent irreducible characters χ_u in $C_{G_\sigma}(t)$ and all r'-elements $x \in T$ the following consequences hold mod πR.

$$\chi_{t,u}(1) \equiv \varepsilon_{C(t)} \frac{1}{|C_{W(T)}(t)|} \varepsilon_T R_{T,\hat{t}}(1) = \frac{|G_\sigma : T|_{p'}}{|C_{W(T)}(t)|},$$

$$\chi_{t,u}(x) \equiv \frac{\varepsilon_{C(t)}\varepsilon_T}{|C_{W(T)}(t)|} R_{T,\hat{t}}(x).$$

Here we denote $C_{G_\sigma}(t)$ and $C_{G_\sigma}(x)$ by $C(t)$ and $C(x)$, respectively. Hence, using Lemma 3.2, one obtains

$$\begin{aligned}
\frac{|x^{G_\sigma}|\chi_{t,u}(x)}{\chi_{t,u}(1)} &\equiv \frac{|G_\sigma : C(x)|\varepsilon_{C(t)}\varepsilon_T R_{T,\hat{t}}(x)|C_{W(T)}(t)|}{|C_{W(T)}(t)|\,|G_\sigma : T|_{p'}} \\
&\equiv \frac{|G_\sigma : C(x)|_p \varepsilon_{C(t)}\varepsilon_T R_{T,\hat{t}}(x)}{|C(x) : T|_{p'}} \\
&\equiv \frac{|G_\sigma : C(x)|_p \varepsilon_{C(t)}\varepsilon_{C(x)}|C(x)|_{p'}\mathcal{N}_{T,\hat{t}}(x)}{|C(x) : T|_{p'}|T|} \\
&\equiv \frac{|G_\sigma : C(x)|_p}{|T|_p}\varepsilon_{C(x)}\varepsilon_{C(t)}\mathcal{N}_{T,\hat{t}}(x) \\
&\equiv \mathcal{N}_{T,\hat{t}}(x)
\end{aligned}$$

because $T \in \{T_0, T_3, T_4, T_6\}$, and therefore $\varepsilon_{C(x)}\varepsilon_{C(t)} = 1$, by Proposition 2.2. Since the right-hand side is independent of the unipotent irreducible character χ_u of $C(t)$, it follows that χ_{t,u_1} and χ_{t,u_2} belong to the same r-block B of G_σ, whenever χ_{u_1} and χ_{u_2} are two unipotent irreducible characters of $C(t)$. Furthermore, $\mathcal{N}_{T,\hat{t}}(x) \equiv \mathcal{N}_{T,\hat{z}}(x)$, because $R_{T,\hat{t}} = R_{T,\widehat{zy}}$ and $R_{T,\hat{z}}$ agree on all r'-elements $x \in T$. Now $\chi_{t,u} \in B$ implies that $\mathcal{N}_{T,\hat{z}}(x) \equiv \mathcal{N}_{T,\hat{s}}(x)$ mod πR. Since D is in the kernels of $\hat{s}$ and $\hat{z}$, and since D is the Sylow r-subgroup of T, it follows that z and s are $W(t)$-conjugate.

For every fixed G_σ-conjugacy class y^{G_σ} of G_σ meeting D let $sy_{G_\sigma} = \{\chi_{sy,u}\}$, where χ_u runs through all the unipotent irreducible characters of $C = C_{G_\sigma}(sy)$, and where $\hat{s}$ denotes the canonical character of the root b of B in $T = C_{G_\sigma}(D)$. Observe that by Proposition 2.2 $C_{G_\sigma}(sy)$ does not contain any unipotent irreducible characters χ_u if and only if sy is a regular element of T; in this case $(sy)_{G_\sigma}$ consists only of the irreducible character $\chi_{sy} = \pm R_{T,\widehat{sy}}$ of G_σ. The above argument with zy replaced by sy shows that for a fixed sy all $\chi_{sy,u} \in \mathrm{Irr}_S(B)$, where χ_u runs through all unipotent irreducible characters of $C = C_{G_\sigma}(sy)$. Thus we have shown that $\mathrm{Irr}_S(B) = \bigcup_{y \in_{G_\sigma} D} (sy)_{G_\sigma}$.

Let y be a representation of a conjugacy class y^{G_σ} with $y^{G_\sigma} \cap D \neq \varnothing$. If sy is not regular, then Proposition 2.2 and the character tables of [**6** and **13**] imply that all unipotent irreducible characters χ_u of $C = C_{G_\sigma}(sy)$ belong to the principal r-block of C. Hence the irreducible characters $\chi_u sy$ of C belong to an r-block β of C with root b in $T = C_C(D)$, because $\hat{s}$ is a canonical character of β. In particular, $B = \beta^{G_\sigma}$ by Brauer's extended first main theorem. If sy is regular, then $b = \{sy \mid y \in D\} = \beta$ and $C_{G_\sigma}(sy) = T$. Hence $B = \beta^{G_\sigma}$. This completes the proof of (d).

(f) By (d) we know that B determines up to G_σ-conjugacy a unique r'-element $s \neq 1$ of G_σ representing the canonical character $\hat{s}$ of B in $T = C_{G_\sigma}(D)$. The number of G_σ-conjugacy classes of maximal tori T_i containing s equals by Proposition 2.2 the number $|s_{G_\sigma}|$ of unipotent irreducible characters χ_u of $C_{G_\sigma}(s)$. By Lemma 5.5 and Theorem 4.3 each irreducible character $\chi_{t,u}$ of B restricted to the r'-elements is a linear combination of the $R_{T_i,s}$. Hence $l(B) = |s_{G_\sigma}|$.

THEOREM 5.9. *Let B be an r-block of $G_\sigma = {}^3D_4(g)$ with defect group $\delta(B) =_{G_\sigma} D \neq 1$, where the prime r does not divide q. Then the following assertions hold*:

(a) *$C = DC_{G_\sigma}(D)$ contains a maximal torus T such that $H = N_{G_\sigma}(D) \leqslant N_{G_\sigma}(T)$.*

(b) *Up to G_σ-conjugacy there exists a unique r'-element $s \in T$ and a root b of B in $C = DC_{G_\sigma}(D)$ such that the linear character $\hat{s}$ of T is the canonical character of B.*

(c) *If $T_H(b)$ denotes the inertial subgroup of b in H, then $T_H(b)/C \simeq C_{W(T)}(s)$, where $W(T) = N_{G_\sigma}(T)/T$.*

(d) *B is the principal r-block of G_σ if and only if $s = 1$ and D is a Sylow r-subgroup of G_σ.*

(e) *An irreducible character $\chi_{t,u}$ of G_σ belongs to B if and only if $t \sim_{G_\sigma} sy$ for some $y \in D$, and χ_u is a unipotent irreducible character of $C_{G_\sigma}(sy)$ such that $sy\chi_u$ belongs to an r-block β of $C_{G_\sigma}(sy)$ with $B = \beta^G$.*

(f) *The number $l(B)$ of modular irreducible characters of B equals the number of unipotent irreducible characters of $C_{G_\sigma}(S)$, provided $s \neq 1$.*

PROOF. If the defect group D of B is cyclic or abelian, then all assertions hold by Propositions 5.6 and 5.8. Hence we may assume that D is not abelian. Thus $r \in \{2, 3\}$ by Lemma 5.2. The proof of (f) is the same as in Proposition 5.8.

Suppose that $r = 3$, and that $3 \mid q - 1$. Then $q \neq 2$. By Proposition 5.4 D is a Sylow 3-subgroup of G_σ. Theorem 9.2 of [**7**, p. 231] asserts that there is a central element $1 \neq x \in D$ of order 3 and a 3-block b_1 of $C_1 = C_{G_\sigma}(x)$ such that $B = b_1^{G_\sigma}$, and b_1 has defect group $\delta(b) = D$. From Proposition 2.2 and the proof of Proposition 5.4 it follows that $C_1 = U \times Z$, where Z is a cyclic group of order $k = \frac{1}{3}(q^2 + q + 1)$, and where U is a nonsplit extension of $\mathrm{SL}_3(q)$ by a cyclic group C_1/C_1' of order 3. In particular, each block b_1 is of the form $b_1 = b_0 \otimes z$, where z denotes a 3-block of defect zero of Z, and b_0 denotes the principal 3-block of U. Because of the structure of C_1 we have $C = DC_{G_\sigma}(D) \leqslant C_1$, and C contains a maximal torus $T \geqslant Z$ such that $T =_{G_\sigma} T_0$, as $3 \mid q - 1$. Now D normalizes T by Corollary 5.19 of [**1**, p. 212]. Since $D \cap T$ is a Sylow 3-subgroup of T and also the largest abelian normal subgroup of D, it follows from Propositions 1.2 and 2.2 that $H = N_{G_\sigma}(D) \leqslant N_{G_\sigma}(T)$.

The 3-block z of Z consists of one linear character $\hat{s}$ of $\hat{T}$, because $Z \leqslant T$. As $Z \leqslant C = DC_{G_\sigma}(D_1) \leqslant C_1$ and $b_1 = b_0 \otimes z$, it follows from Brauer's extended first main theorem that $\hat{s}$ is the canonical character of a common root block b of B and b_1 in C. Certainly

$$T_H(b) = \left\{h \in N_{G_\sigma}(D) \mid s^h = s\right\} = \left\{h \in N_{G_\sigma}(T) \mid s^h = s\right\}$$

by Proposition 1.2. Therefore $T_H(b)/C \simeq C_{W(T)}(s)$.

Since $b_1 = b_0 \otimes z$, and $z = \{\hat{s}\}$, it follows from Brauer's third main theorem that $B = B_0$ is the principal 3-block of G_σ if and only if $s = 1$ and D is a Sylow 3-subgroup of G_σ.

As shown in the proof of Proposition 5.4, the cyclic group C_1/C_1' of order 3 acts trivially on Z. Thus $y \in C_{G_\sigma}(s)$ for every $y \in D$.

Let χ_u be a fixed unipotent irreducible character of $C_y = C_{G_\sigma}(sy)$. Now $D \cap T'$ is a Sylow 3-subgroup of T' for every maximal torus $T' \subseteq C_y$ containing sy. Therefore the irreducible characters $\chi_{sy,u}$ of G_σ agree on all 3'-conjugacy classes of G_σ by the proof of Lemma 5.5 and Theorem 4.3, because $D \cap T' \subseteq \ker \hat{s}$ and $s \in T'$. Hence Osima's theorem and Lemma 4.2 of [7, p. 150], imply that all $\chi_{sy,u}$ with $y \in D$ belong to the same 3-block B' of G_σ. By Proposition 2.2 the unipotent irreducible characters χ_u of C_y belong to the principal 3-block b_0^* of C_y. Hence by the structure of C_y the irreducible characters $\widehat{sy}\chi_u$ belong to one 3-block b_y of $C_y = C_{G_\sigma}(sy)$ with defect group $D_2 = D \cap C_y$ and canonical character $\hat{s}$. As $C_{C_y}(D_2) \subseteq C_y$, Lemma 6.1 of [7, p. 209] asserts the existence of $(b_y)^{G_\sigma}$, and $B = (b_y)^{G_\sigma}$ by Brauer's extended first main theorem, because both blocks have the same canonical character. Applying now the proof of Proposition 5.8(d) we see that $B' = B$. Hence all irreducible characters $\chi_{sy,u}$ of G_σ such that $\widehat{sy}\chi_u$ belongs to a 3-block b_y of $C_{G_\sigma}(sy)$ with $(b_y)^{G_\sigma} = B$ are contained in B.

As $3 \mid q-1$, it follows from Theorem 4.3 and Table 4.4 that B_0 contains the unipotent irreducible characters

$$U(B_0) = \left\{1, [\varepsilon_1], [\varepsilon_2], \mathrm{St}, \rho_1, \rho_2, {}^3D_4[1]\right\}.$$

Since the order of the Sylow 3-subgroup P divides the degree of ${}^3D_4[-1]$, this unipotent irreducible character belongs to a 3-block of G_σ with defect zero. Hence all other 3-blocks of G_σ with positive defect are not unipotent.

For every fixed G_σ-conjugacy class y^{G_σ} of G_σ meeting D let $sy_{G_\sigma} = \{\chi_{sy,u}\}$, where χ_u runs through all the unipotent irreducible characters of $C_y = C_{G_\sigma}(sy)$. Let $y_1 = 1$, $y_2, \dots, y_t$ be representatives of these conjugacy classes of 3-elements. As no y_i is conjugate to the involution s_2 it follows from Proposition 2.2 that $C_{y_i} = C_{G_\sigma}(sy_i) = C_{G_\sigma}(y_i)$ for $i = 2, 3, \dots, t$ provided sy_i is of type s_4 or s_5 and $s \neq 1$ or $s = 1$ and y_i is of type s_3, s_4, or s_5. In particular, the irreducible characters $\widehat{sy_i}\chi_u$ of C_{y_i} belong to one 3-block b_{y_i} of $C_{y_i} = C_{G_\sigma}(y_i) = C_G(sy_i)$ with $B = (b_{y_i})^{G_\sigma}$, and the number of 3-modular characters of b_{y_i} is $l(b_{y_i}) = |(sy_i)_{G_\sigma}|$ for $i = 2, 3, \dots, t$, because no sy_i is regular by Propositions 2.1 and 2.2 and Table 4.4. An application of Theorem 68.4

of [6] now yields that the number of ordinary irreducible characters of B is

$$k(B) = \sum_{i=1}^{t} l(b_{y_i}) = \sum_{i=1}^{t} |(sy_i)_{G_\sigma}|.$$

This completes the proof of (e) in the case $r = 3$ and $3 \mid q - 1$.

If $3 \mid q + 1$, then s_3, s_4, and s_5 are replaced by the representatives s_7, s_9, and s_{10}, respectively. Furthermore, it follows from Theorem 4.3 and Table 4.4 that the principal 3-block B_0 contains the unipotent irreducible characters $U(B_0) = \{1, [\varepsilon_1], [\varepsilon_2], \mathrm{St}, \rho_2, {}^3D_4[-1], {}^3D_4[1]\}$, and in this case ρ_1 is of defect zero. With these changes the above argument applies in this case. Hence Theorem 5.9 holds for $r = 3$.

So we may assume that $r = 2$, and q is odd. With the notation of Proposition 5.3 it follows that D is one of the 2-groups P, $S*Z$, or S, where P is a Sylow 2-subgroup of G_σ, S is a semidihedral group of order $|S| = 2^{a+2}$ and Z is a cyclic group of order $|Z| = 2^a$. Furthermore, by Propositions 1.2, 2.1, and 5.3 we may assume that $q \equiv 1 \bmod 4$, because the case $q \equiv 3 \bmod 4$ follows similarly.

Suppose that D is a semidihedral group of order $|D| = 2^{a+2}$, where 2^a is the highest power of 2 dividing $q - 1$. As D is a defect group of the 2-block B, Theorem 3.15 of [12] and the proof of Proposition 5.3 imply that $k(B) = 2^a + 4$, $k_0(B) = 4$, $k_1(B) = 2^a - 1$ and $k_v(B) = 1$, where $k_i(B)$ denotes the number of irreducible characters of B with highest i, and where $v = 2^a$. By Theorem 9.2 of [7, p. 231], there is a central element $1 \neq x \in D$ and a 2-block b_1 of $C_1 = C_{G_\sigma}(x)$ such that $B = b_1^{G_\sigma}$ and D is a defect group of b_1. Again by the proof of Proposition 5.3 we may assume that x is either of type s_7 or s_{10}. Let x be of type s_7. Then by Propositions 1.2 and 2.1 C_1 contains a maximal torus $T_1 \simeq \mathbf{Z}_{(q^3-1)(q+1)}$ such that $C = DC_{G_\sigma}(D) \geqslant T_1$. Furthermore, there exists up to G_σ-conjugacy a unique element $s \in T_1$ of odd order dividing $q + 1$ such that the linear character $\hat{s}$ of T_1 is the canonical character of a common root block b of C of the blocks B and b_1. Each irreducible character $\chi_{sy,u}$ of G_σ with $y \in D$ belongs to B by the proof of Proposition 5.8(d), Lemma 5.5, and Theorem 4.3. The center $Z(D)$ of D has order 4. Applying Propositions 1.2 and 2.2 and Lemma 3.4 we see that there are two conjugacy classes of G_σ of the form sy with $y \in Z(D)$. As D is a Sylow 2-subgroup of $C_{G_\sigma}(sy)$ for $y \in Z(D)$ it follows from Proposition 2.2 and Table 4.4 that each of the four irreducible characters $\chi_{sy,u}$ with $y \in Z(D)$ has height zero. Since $k_0(B) = 4$, all other irreducible characters of B have positive height. By Proposition 1.2 T_1 has a cyclic Sylow 2-subgroup $\langle y \rangle$ of order 2^{a+1}. Therefore $y^i \notin Z(D)$ for $1 \leqslant i \leqslant 2^a - 1$. Proposition 2.1 and Table 4.4 assert that each element sy^i of T_1 is regular. Thus each irreducible character χ_{sy^i}, $1 \leqslant i \leqslant 2^a - 1$, of B has height 1 by Table 4.4. As $q \equiv 1 \bmod 4$ the Sylow 2-subgroup of the maximal torus T_6 of G_σ is a Klein four group by Proposition 1.2. Applying again Table 4.4 and Proposition 2.1, we see that there is a $y \in D$ such that sy is a regular element of T_6. Hence χ_{sy} is an irreducible character of B with height $v = 2^a$. Therefore we have found all irreducible characters of B. Replacing T_1 and s_7 by T_2 and s_{10}, respectively, the remaining case is proved similarly. Hence all assertions (a)–(e) hold for blocks B of G_σ with a semidihedral defect group D, because the same arguments hold in the case $q \equiv 3 \bmod 4$.

Suppose that $D \simeq S * Z$, where S is a semidihedral group of order $|S| = 2^{a+2}$ and Z is a cyclic group of order $|Z| = 2^a$. Let $q \equiv 1 \bmod 4$. By Theorem 9.2 of [7, p. 231], there is a central element $1 \neq x \in D$ and a 2-block b_1 of $C_1 = C_{G_\sigma}(x)$ such that $B = b_1^{G_\sigma}$ and D is a defect group of b_1. Again by proof of Proposition 5.3 we may assume that x is either of type s_3 or s_5. In both cases it follows that C_1 contains a maximal torus $T_0 \simeq Z_{q^3-1} \times Z_{q-1}$ such that $C = DC_{G_\sigma}(D) \geqslant T_0$. Let x be of type s_3, and let b be a root of B and therefore of b_1 in C. Then there exists up to G_σ-conjugacy a unique element $s \in T_0$ of odd order dividing $q - 1$ and of type s_3 such that the linear character $\hat{s}$ of T_0 is the canonical character of B. Applying again Lemma 5.5, Theorem 4.3, and the proof of Proposition 5.8(d) it follows that each irreducible character $\chi_{sy,u}$ of G_σ with $y \in D$ belongs to B. Using now Proposition 2.2 and Theorem 68.4 of [6] as in the case $r = 3$ it follows that we have found all irreducible characters of B. The remaining cases $x \sim_{G_\sigma} s_5$ and $q \equiv 3 \bmod 4$ are dealt with similarly.

By Proposition 5.3 only the principal 2-block B_0 of G_σ has a Sylow 2-subgroup P as a defect group. Furthermore, the above argument shows that each irreducible character $\chi_{y,u}$ with $y \in P$ belongs to B_0. Therefore all unipotent irreducible characters of G_σ are contained in B_0. This completes the proof of Theorem 5.9.

As a first application of Theorem 5.9 we verify Brauer's height zero conjecture in the case of the simple triality groups $G_\sigma = {}^3D_4(q)$.

COROLLARY 5.10. *Let B be an r-block of G with defect group D. Then every irreducible character χ of G_σ belonging to B has height zero if and only if D is abelian.*

PROOF. If $r \mid q$, then by Humphreys [10] we may assume that $B = B_0$, the principal r-block of G_σ. The Sylow r-subgroup of G_σ has order q^{12} and is not abelian. By Table 4.4, B_0 has unipotent irreducible characters of positive height. So we may suppose that $r \nmid q$.

By Lemma 5.2 and Propositions 5.3 and 5.4 the r-block B has an abelian defect group D if and only if D is a Sylow r-subgroup of a maximal torus T of G_σ. Hence, if D is abelian, then Theorem 5.9 and Table 4.4 imply that all irreducible characters of B have height zero. Suppose that D is not abelian. Then $r \in \{2,3\}$ by Lemma 5.2. By Theorem 5.9(a) and (b) $C = DC_{G_\sigma}(D)$ contains a maximal torus T of G_σ such that there is up to G_σ-conjugacy a unique r'-element $s \in T$ which corresponds to the canonical character of B. In the proof of Theorem 5.9(e) we have shown that for every $y \in D$ which is not G_σ-conjugate to a central element of D the irreducible characters $\chi_{sy,u}$ of B have positive height. This completes the proof.

Brauer's conjecture on the number $k(B)$ of irreducible characters of an r-block B of G_σ follows also.

COROLLARY 5.11. *Let B be an r-block of G_σ with defect group D. Then $k(B) \leqslant |D|$.*

PROOF. Since $k(B) = 1$ for every r-block of defect zero, we may assume that $|D| \neq 1$.

If $r \mid q$, then B is the principal block of G_σ by [10], and

$$k(B) = \begin{cases} q^4 + q^3 + q^2 + q + 4, & \text{if } 2 \mid q, \\ q^4 + q^3 + q^2 + q + 5, & \text{if } 2 \nmid q, \end{cases}$$

by Proposition 2.3. As $|D| = q^{12}$, we get $k(B) \leqslant |D|$.

Suppose that $r \nmid q$. If D is abelian, then D is a Sylow r-subgroup of a maximal torus T of G_σ by Lemma 5.2, Proposition 5.3, and Proposition 5.4. Therefore Proposition 1.2 asserts that D can be generated by one or two elements. Thus $k(B) \leqslant |D|$ by Theorem 10.13 of [7, p. 316].

If D is nonabelian, then $r \in \{2, 3\}$ by Lemma 5.2. Let $r = 3$. Then $3 \nmid q$, and D is a Sylow 3-subgroup of G_σ by Proposition 5.4. Suppose that 3^a is the highest power of 3 dividing $q - 1$. By Theorem 5.9 there is a semisimple $3'$-element s of G_σ such that each irreducible character χ of B is of the form $\chi = \chi_{sy,u}$, where y is G_σ-conjugate to an element of D, and where χ_u is a unipotent irreducible character of $C_{G_\sigma}(sy)$. Furthermore, $B = B_0$ if and only if $s = 1$. Since by the proof of Theorem 5.9 the principal 3-block B_0 has 7 unipotent irreducible characters, it follows from Proposition 2.2 and Theorem 5.9(e) that

$$k(B) = \begin{cases} 6 + 4 \cdot 3^a, & \text{if } s \neq 1, \\ 10 + 4 \cdot 3^a, & \text{if } s = 1. \end{cases}$$

In any case $k(B) \leqslant 3^{2+2a} = |D|$. The same argument applies, if $3 \mid q + 1$.

Let $r = 2$. Then $2 \nmid q$, and $D \in \{P, S * Z, S\}$ by Proposition 5.3, where P is a Sylow 2-subgroup of G_σ of order $|P| = 2^{2+2a}$, S is a semidihedral group of order $|S| = 2^{a+2}$, and Z is a cyclic group of order $|Z| = 2^a$, and where 2^a is the highest power of 2 dividing $q - 1$ or $q + 1$. Furthermore, the principal 2-block B_0 is the only 2-block of highest defect, and it has 8 unipotent irreducible characters by the proof of Theorem 5.9(e). Another application of Proposition 2.2, Table 4.4, and Theorem 5.9(e) yields that for $2^a \mid q - 1$

$$k(B) = \begin{cases} 14 + 2^a & \text{if } D =_{G_\sigma} P \text{ and } q = 3, \\ 14 + 2^{a+1} & \text{if } D =_{G_\sigma} P \text{ and } q \neq 3, \\ 2 + 2^{a+1} & \text{if } D =_{G_\sigma} S * Z, \\ 4 + 2^a & \text{if } D =_{G_\sigma} S. \end{cases}$$

Hence $k(B) \leqslant |D|$ in any of these cases. If $2^a \mid q + 1$, then the assertion follows by a similar count for $k(B)$. This completes the proof.

We also can verify the Alperin-McKay conjecture for the simple triality groups.

COROLLARY 5.12. *Let B be an r-block with defect group D of G_σ. Let B_1 be the Brauer correspondent of B in $H = N_{G_\sigma}(D)$. Then $k_0(B) = k_0(B_1)$.*

PROOF. If $r = p \mid q$, then by [9] $k_0(B) = k_0(B_1)$, as was pointed out by Feit [7, p. 171].

Let $r \nmid q$. If D is abelian, then $k(B) = k_0(B)$ by Corollary 5.10. Furthermore, $k(B) = k(B_1)$ by Propositions 5.6 and 5.8 and Lemmas 3.4 it and 3.5. Using the proof of Lemma 5.7 it follows that $k_0(B_1) = k(B)$. Hence $k_0(B) = k_0(B_1)$.

Let D be nonabelian. If $r = 2$, then by the proof of Theorem 5.9 and Corollary 5.10 it follows (with the notation of Proposition 5.3) that

$$k_0(B) = k_0(B_1) = \begin{cases} 8 & \text{if } D =_{G_\sigma} P, \\ |Z| & \text{if } D =_{G_\sigma} S * Z, \\ 4 & \text{if } D =_{G_\sigma} S. \end{cases}$$

So we may assume that $r = 3$. As D is not abelian, it is a Sylow 3-subgroup of G_σ by Proposition 5.4.

If $q = 2$, then G_σ has only the principal 3-block B_0 as a 3-block of highest defect. By Table 4.4 and the proof of Theorem 5.9 the set of irreducible characters of B_0 with height zero is

$$\mathrm{Irr}^0(B_0) = \left\{1, [\varepsilon_1], [\varepsilon_2], \mathrm{St}, {}^3D_4[-1], {}^3D_4[1], \chi_{9,1}, \chi_{9,\mathrm{St}}, \chi_{9,qs'}\right\},$$

where s_9 denotes a representative of order 3. As $q = 2$ it follows from Proposition 1.2, Lemma 3.5, and Theorem 5.9 that $H = N_{G_\sigma}(D) = N_{G_\sigma}(T_6)$. Therefore $H = U_3(2)$ by Proposition 2.2. Let b_0 be the principal 3-block of H. Using the character table of Simpson and Frame [**1**] it follows that $\mathrm{Irr}^0(b_0) = \{1, 1_1, 1_2, 2, 2_2, 8, 8_1, 8_2\}$, where the irreducible characters of H with height zero are denoted by their degrees. As $B_1 = b_0$ by Brauer's third main theorem, we obtain that $k_0(B) = 9 = k_0(B_1)$.

Thus we may assume that $q > 2$. Hence either $3 \mid q - 1$ or $3 \mid q + 1$. Let $3 \mid q - 1$. Since the number of 3-blocks with highest defect equals the number of 3-regular conjugacy classes with highest defect, it follows from Table 4.4 and Proposition 2.2 that G_σ has the principal 3-block B_0 and $\frac{1}{3}(q^2 + q - 2)$ many nonunipotent 3-blocks B with defect group $\delta(B) =_{G_\sigma} D$. Let $B = B_0$. By the proof of Theorem 5.9

$$\mathrm{Irr}^0(B_0) = \left\{1, [\varepsilon_1], [\varepsilon_2], \mathrm{St}, \rho_1, \rho_1, \chi_{4,1}, \chi_{4,\mathrm{St}}, \chi_{4,qs}\right\},$$

where s_4 denotes a representative of order 3. From Theorem 5.9, Proposition 1.2 and Lemma 3.4, it follows that $H = N_{G_\sigma}(D) = N_{G_\sigma}(T_0)$ and $H/T_0 \simeq D_{12}$. Using the action of D_{12} on T_0 it is easy to see that the Brauer correspondent B_1 of B_0 has $k_0(B_1) = k_0(b_0) = 9$. Thus $k_0(B_0) = 9 = k_0(B_1)$.

Now let B be a nonprincipal 3-block, and b its Brauer correspondent in $H = N_{G_\sigma}(D) = N_{G_\sigma}(T_0)$. Then $\mathrm{Irr}^0(B) = \{\chi_{4,1}, \chi_{4,\mathrm{St}}, \chi_{4,qs}\}$, where $s_4 = yc$ with y an element of order 3 in the center of $\mathrm{SL}_3(q)$ and $c \neq 1$ a fixed representative of 3′-conjugacy class of the cyclic group $S_\sigma = \mathbf{Z}_{q^2+q+1}$ described in Proposition 2.2. Also b is determined by the conjugacy class c^H, as follows from Brauer's first main theorem. From Theorem 5.9(c) and Lemma 3.4 it follows that $k_0(b) = k_0(B)$.

The remaining case $3 \mid q + 1$ follows similarly, with q replaced by $(-q)$, which means replacing s_4 by s_9 and ρ_1, ρ_2 by ${}^3D_4[-1]$, ${}^3D_4[1]$, respectively. This completes the proof.

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